Spaces of Modular Symbols

Karim Belabas, Bernadette Perrin-Riou

after Pollack and Stevens

Institut de Mathématiques de Bordeaux

Modular forms

Let $G \subset \mathrm{PSL}(2,\mathbb{Z})$ be a subgroup of finite index, and V be a right G-module. We are interested in the cohomology of the modular curve $X(G)$ with coefficients in V, more precisely in $H^1_c(X(G),V)$, as a Hecke-module.

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Standard example: $G = \Gamma_0(N), V = \mathrm{Sym}^{k-2}\,\mathbb{C}^2$ (Shimura),

$$(P \mid \gamma)(X,Y) = P(dX - cY, -bX + aY), \quad P \in V.$$

We recover classical $\mathbb{C}$-vector spaces of holomorphic modular forms for G:

$$H^1_c(X(G),V) \simeq S_k(G) \oplus S_k(G) \oplus E_k(G)$$

Another interesting example

Let Γ be a congruence subgroup of level prime to p and $G = \Gamma \cap \Gamma_0(p)$; let $V = \mathcal{D}_k(\mathbb{Z}_p) =: \mathcal{D}$, the space of locally analytic p-adic distributions on $\mathbb{Z}_p$, with weight $k-2$ action of G. This specializes via p-adic periods $\rho_k\colon \mathcal{D} \to \operatorname{Sym}^{k-2} \mathbb{Q}_p^2$ via

$$\rho_k\colon \mu \mapsto \int (Y - tX)^{k-2} d\mu(t).$$

This space $\mathcal{D}$ contains the p-adic L-functions μ_f associated to normalized eigenforms $f \in S_k(\Gamma)$, and satisfying interpolation properties relating $\mu_f(t^j \cdot \chi)$ and special values $L(f, \chi^{-1}, j+1)$, where χ is of finite order on $\mathbb{Z}_p^\times$.

This allows to define specializations

$$\operatorname{Symb}_G(\mathcal{D}) \to \operatorname{Symb}_G(\operatorname{Sym}^{k-2} \mathbb{Q}_p^2)$$

The target of this map is finite dimensional while the source has infinite dimension. Nevertheless, by restricting to natural subspaces, Pollack and Stevens obtain a Hecke-equivariant isomorphism.

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Let $\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$, generated by the divisors $[\beta] - [\alpha]$, which we denote by $\{\alpha, \beta\}$ and see as a path through the completed upper half plane $\overline{\mathcal{H}}$ linking the two cusps $\alpha \to \beta$. This is a left $\mathrm{GL}(2, \mathbb{Q})$-module via fractional linear transformations:

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Finally

$$\mathrm{Symb}_G(V) := \mathrm{Hom}_G(\Delta_0, V), \quad \phi \mid \gamma = \phi, \forall \phi \in G.$$

(V-valued modular symbols on G)

Theorems

Theorem (Ash-Stevens). *Provided that the orders of torsion elements of* Γ *act invertibly on* V *(e.g. if* V *is a vector space), we have a canonical isomorphism*

$$\mathrm{Symb}_G(V) \simeq H^1_c(X(G), V).$$

Assume V also allows a right action by $\mathrm{GL}(2,\mathbb{Q})$, then we can define a Hecke action on $\mathrm{Symb}_G(V)$. If ℓ is prime then T_ℓ is given by the double coset $G\left(\begin{smallmatrix}1&0\\0&\ell\end{smallmatrix}\right)G$. E.g. if $G=\Gamma_0(N)$ and $\ell\nmid N$, then

$$\phi \mid T_\ell = \phi \mid \left(\begin{smallmatrix}\ell&0\\0&1\end{smallmatrix}\right) + \sum_{a=0}^{\ell-1} \phi \mid \left(\begin{smallmatrix}1&a\\0&\ell\end{smallmatrix}\right).$$

If $\sigma := \left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ normalizes G, then it acts as an involution on $\mathrm{Symb}_G(V)$; if 2 acts invertibly on V, this yields the expected decomposition

$$\mathrm{Symb}_G(V) \simeq \mathrm{Symb}_G(V)^+ \oplus \mathrm{Symb}_G(V)^-$$

into eigenspaces for this action.

Computation ?

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ and $B = [\Gamma : G]$. Assume that

- $G\backslash \mathrm{PSL}(2,\mathbb{Z})$ can be enumerated in time $\widetilde{O}([\Gamma : G])$, with representatives of size $O(\log B)$.
- equivalence $\gamma_1 \sim_G \gamma_2$, together with $\gamma \in G$ realizing it, is tested in time $\mathcal{O}(\log B)^C$ if $\|\gamma_j\| = O(B)$,

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Then the structure of Δ_0 as a G-module is computed in time $\widetilde{O}(B)$, by writing down a nice fundamental domain for G (connected boundary, all edges are unimodular paths):

- minimal system of generators (g_i), $i \leqslant n$, $g_n = \{0,\infty\}$.
- relations explicitly written down (without computation),
- solve discrete logs: given a path p, find γ_i in $\mathbb{Z}[G]$ such that $p = \sum_i \gamma_i \cdot g_i$.

Computation ?

Among the n generators g_i, we get

- one relation for each conjugacy class of 2-torsion elements in G: $(1+\gamma_i)\cdot g_i = 0$, $1 \leqslant i \leqslant s$
- one for each pairs of conjugacy classes of 3-torsion elements: $(1+\gamma_i+\gamma_i^2)\cdot g_i = 0$, $s+1 \leqslant i \leqslant s+r$.
- and one "boundary relation" (walk around the fundamental domain and come back to starting point).

These generate all relations.

Corollary . *Given G a finite index subgroup and V a right G-module. Choose any $n-1$ elements $v_i \in V$, compatible with the torsion relations when $i \leqslant s+r$ (e.g. $v_i(1+\gamma_i) = 0$, i.e restrict v_i to an eigenspace $V_i \subset V$). Solve for v_n so that the boundary relation is satisfied. Then $\phi(g_i) = v_i$ defines a modular symbol ϕ, and all modular symbols arise in this way.*

Implementation in `libpari`

Adapted from the SHP package by Robert Pollack. Currently only for $G = \Gamma_0$ and $V = \mathrm{Sym}^{k-2}\,\mathbb{Q}^2$.

- structure of Δ_0 as a G-module,
- explicit $\mathbb{Q}$-basis for $\mathrm{Symb}_G(V)$, dimension $d \approx kN/6$,
- $\sigma \in M_d(\mathbb{Q}) \Rightarrow \mathrm{Symb}_G(V)^{\pm}$,
- Hecke operators T_ℓ ($\ell \nmid N$) and U_ℓ ($\ell \mid N$),
- Eisenstein subspace E of $\mathrm{Symb}_G(V)^{\pm}$ ($\mathbb{Q}$-basis),
- Cuspidal subspace S of $\mathrm{Symb}_G(V)^{\pm}$: Hecke stable supplement of E, such that eigenvalues don't mix, i.e. $\gcd(\mathrm{char}(T_p \mid S), \mathrm{char}(T_p \mid E)) = 1$,
- Degeneracy maps $\mathrm{Symb}_{\Gamma_0(N/p)}(V) \to \mathrm{Symb}_{\Gamma_0(N)}(V)$, whose image give S^{old}. Restrict to S and compute a Hecke-stable supplement $\Rightarrow S^{\text{new}}$,
- Decomposition of S^{new} into simple subspaces.

Implementation in `libpari`

- Cut out a modular symbol with given system of Hecke-eigenvalues
  $\Rightarrow E/\mathbb{Q} \to \phi_E \in \mathrm{Symb}_{\Gamma_0(N_E)}(\mathbb{Q})$.