

TUTORIAL: ELLIPTIC CURVES OVER FINITE FIELDS IN PARI/GP

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This is a mostly self-contained introduction to computations in curves over finite fields in PARI/GP. PARI/GP is free software computer algebra system available at

<http://pari.math.u-bordeaux.fr/download.html>

In the following, `?` denote the GP prompt and `\\` denote GP comments. A text version of the list of commands to enter is available at http://ecc2015.math.u-bordeaux.fr/documents/ecc_gp.txt

Explanation of the syntax is in the appendix.

It is recommended to use PARI/GP 2.7.x.

Please do

```
? \\1 ecc.log
```

to enable logging of your input for future reference.

1. PRIME FINITE FIELDS

To create a random prime number:

```
? p=randomprime(2^100)
```

```
%1 = 792438309994299602682608069491
```

To create an element of \mathbb{F}_p :

```
? a=Mod(2,p)
```

```
? a^(p-1) \\ powering
```

```
%3 = Mod(1,792438309994299602682608069491)
```

2. GENERAL FINITE FIELDS

To build an irreducible polynomial of degree n of \mathbb{F}_p , use `ffinit(p,n)`.

```
? P=ffinit(13,2)
```

```
%4 = Mod(1,13)*x^2+Mod(1,13)*x+Mod(12,13)
```

```
? polisirreducible(P)
```

```
%5 = 1
```

To build an element of \mathbb{F}_{p^n} (also work for $n = 1$) from its minimal polynomial:

```
? a=ffgen(P,'a)
```

The above can be abbreviated by `ffgen(p^n,'a)`

```
? a=ffgen(13^7,'a)
```

Basic operations:

```
? a^10458086 \\ powering
? fforder(a) \\ order of an element
? minpoly(a) \\ minimal polynomial
? random(a) \\ random element of  $F_p^n$ 
```

To get a generator of \mathbb{F}_p^\times :

```
? b = ffprimroot(a)
? fforder(a)
? fforder(b)
```

Discrete logarithm:

```
? n=fflog(a,b)
? b^n
```

2.1. **Exercise.** Compute a discrete logarithm in $\mathbb{F}_{2^{127}}$ and see how much time it take. Use # to activate the timer or ## to see the time of the last command.

3. ELLIPTIC CURVES OVER FINITE FIELDS

From a short Weierstrass model $y^2 = x^3 + a_4x + a_6$:

```
? Es = ellinit([a^4,a^6],a);
```

From a long Weierstrass model $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$:

```
? E = ellinit([a,a^2,a^3,a^4,a^6],a);
```

Basic functions:

```
? E.j \\ j-invariant
```

Structure of the group $E(\mathbb{F}_q)$

```
? ellcard(E) \\ cardinal of  $E(\mathbb{F}_q)$ 
```

```
? ellgroup(E) \\ structure of  $E(\mathbb{F}_q)$ 
```

Above $[d_1, d_2]$ means $\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z}$, with $d_2 \mid d_1$.

```
? G=ellgenerators(E) \\ generators of  $E(\mathbb{F}_q)$ 
```

A minimal generating set.

```
? P = random(E) \\ random point on  $E(\mathbb{F}_q)$ 
```

```
? Q = random(E) \\ another random point on  $E(\mathbb{F}_q)$ 
```

```
? ellisoncurve(E, P) \\ check that the point is on the curve
```

```
? elladd(E, P, Q) \\  $P+Q$  in  $E$ 
```

```
? ellmul(E, P, 100) \\  $100.P$  in  $E$ 
```

```
? oP = ellorder(E,P) \\ order of  $P$ 
```

```
? oQ = ellorder(E,Q) \\ order of  $Q$ 
```

```
? o = lcm(oP,oQ); \\ both  $P$  and  $Q$  are in  $E(\mathbb{F}_q)[o]$ 
```

```
? w=ellweilpairing(E,P,Q,o) \\ Weil pairing of  $P$  and  $Q$  of order  $o$ 
```

```
? fforder(w)
```

```
? nP = ellmul(E, P, random(o));
```

```
? n = elllog(E,nP,P)
```

```
? ellmul(E,P,n) == nP
```

4. APPLICATION: SPEED OF DISCRETE LOGARITHM

We want compare the speed of discrete logarithm over \mathbb{F}_p and $E(\mathbb{F}_p)$, for p of 30, 40 and 45 bits. For fairness, we need to use groups of nearly prime orders.

```
? until(isprime((p-1)/2), p=randomprime(2^30));p
? g=ffprimroot(ffgen(p))^2; a=g^random(p);
? fflog(a,g)
? ##
? until(isprime(ellcard(E)),E=ellinit([1,random(g)],g));
? G=ellgenerators(E)[1]; P=ellmul(E,G,random(ellcard(E)));
? elllog(E,P,G)
? ##
```

5. APPLICATION: THE MOV ATTACK ON THE DISCRETE LOGARITHM PROBLEM

We build curves where the Weil pairing allows us to reduce the discrete logarithm problem on the curve to a discrete logarithm problem on a finite field. This is the idea behind the MOV attack of Menezes, Okamoto, and Vanstone.

5.1. **Example over \mathbb{F}_p .** We will use the curve $E : y^2 = x^3 + x$ over \mathbb{F}_p where $p = 4n^2 + 1$ for some integer n , which has the property that $E(\mathbb{F}_p)$ is isomorphic to $\mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$. We choose n to be prime. To find suitable n of 50 bits:

```
? until(isprime(p),n=randomprime(2^50);p=1+4*n^2);p
? a=ffgen(p); E=ellinit([1,0],a);
? ellgroup(E)
? [P,Q] = ellgenerators(E); \\ name P and Q the generators.
We use the Weil pairing with the second generator to solve the discrete
logarithm in the group generated by the first one:
? e = random(2*n)
? R = ellmul(E,P,e);
? wR = ellweilpairing(E,Q,R,2*n);
? wP = ellweilpairing(E,Q,P,2*n);
? default(parisize,"32M");
? fflog(wR,wP,2*n)
? ##
? elllog(E,R,P,2*n)
? ##
```

5.2. **Example over \mathbb{F}_{p^2} .** We will use the curve $E : y^2 = x^3 + x$ and a prime $p \equiv 3 \pmod{4}$, so that E is supersingular, of order $p + 1$. To solve a discrete logarithm problem in $E(\mathbb{F}_p)$, we embed it in $E(\mathbb{F}_{p^2})$ which is isomorphic to $\mathbb{Z}/(p + 1)\mathbb{Z} \times \mathbb{Z}/(p + 1)\mathbb{Z}$, so we can use the Weil pairing in $E(\mathbb{F}_{p^2})$. We choose $(p + 1)/4$ to be prime.

```

? until(p%4==3 && isprime((p+1)/4),p=randomprime(2^52));
? a=ffgen(p^2,'a);
? E=ellinit([1,0],p); \\ E(F_p)
? ellgroup(E)
? [P] = ellgenerators(E)
? E2=ellinit([1,0],a); \\ E(F_p^2)
? [m,m]=ellgroup(E2)
? [P1,Q] = ellgenerators(E2)
? ellweilpairing(E2,P,Q,m);
? e = random(m)
? R = ellmul(E,P,e);
? wR = ellweilpairing(E2,Q,R,m);
? wP = ellweilpairing(E2,Q,P,m);
? fflog(wR,wP,m)
? ##
? elllog(E,R,P,m)
? ##

```

5.3. Exercises.

- Do the same for a supersingular curve over $\mathbb{F}_{3^{41}}$.
- Write a function that compute the group structure of $E(\mathbb{F}_q)$ using the Weil pairing. See the syntax for functions in the appendix.
- Implement the Pollard rho algorithm to compute discrete logarithm on $E(\mathbb{F}_q)$.

6. TO KNOW MORE ABOUT PARI/GP

See our website <http://pari.math.u-bordeaux.fr> or attend the workshop Atelier PARI/GP 2016 (Jan 11th-15th 2016) at Institut Fourier (Grenoble).

Cheat sheet

APPENDIX A. BASIC INPUT

From a terminal, typing `gp` starts the interpreter.

<code>1+1</code>	basic operation
<code>1+1;</code>	basic operation, no printout
<code>quit</code> or <code>\q</code>	exit the interpreter
<code>?</code>	online help
<code>??function</code>	extended online help about <i>function</i>
<code>#</code>	start timer
<code>read("file")</code> or <code>\r file</code>	load <i>file</i> in interpreter

Line by line evaluation. Enclose multi-line statements between `{...}`.

APPENDIX B. BASIC COMMANDS

<code>/*...*/</code>	comment
<code>a = 1</code>	assignment
<code>[u,v,d] = gcdext(5,8)</code>	simultaneous assignment ($d = 1, u = -3, v = 2$)
<code>a == 1</code>	equality test
<code>%, %5</code>	last result, 5-th result in history
<code> , &&, !</code>	boolean operators (or, and, not)
<code>i++, i--, i+=2</code>	increase i by 1, -1, 2
<code>v[i]</code>	i -th component of vector v
<code>print(x, ":", y)</code>	outputs the value of x , a colon, then y

APPENDIX C. CONSTRUCTORS

<code>random([100,200])</code>	a uniform integer in $[100, 200]$
<code>vector(5, i, 2*i+1)</code>	$\{2i + 1 : 1 \leq i \leq 5\} = [3, 5, 7, 9, 11]$
<code>Mod(2,N)</code>	2 in $\mathbb{Z}/N\mathbb{Z}$
<code>g = ffgen(t^2+Mod(1,3))</code>	t in $\mathbb{F}_3[t]/(t^2 + 1) = \mathbb{F}_9$

APPENDIX D. FLOW CONTROL

```
for(i = 1, 10, print(i)) /* 1,2,...,10 */
i = 8; while(!isprime(i), i++)
until(isprime(p), p = random(1000))

if (isprime(p), print("yes"))
{ if (isprime(p), /*then*/ print("yes")
, /*else*/ print("no")); }
```

APPENDIX E. USER FUNCTIONS

```
f(x,y) =
{ my(z = x + y); /* local variable */
  if (z < 0, return (-1));
  return (1);
}
```