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Sharing a secret key using supersingular isogenies

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Secure key exchange

Diffie-Hellman [DH76]

Given a group $(G, \cdot)$ and a generator of this group g Alice and Bob compute a shared secret proceeding as follows :

- Alice (resp. Bob) computes $h_a = g^a$ ($h_b = g^b$) from her (resp. his) secret exponent a (b) and sends it ;

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Now we will do the same with supersingular isogeny following [LDJ14, JAC+17].

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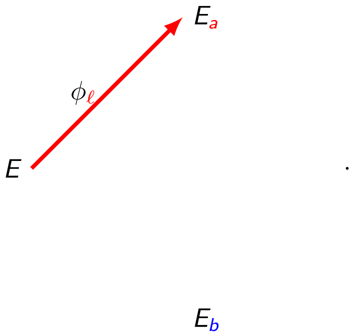

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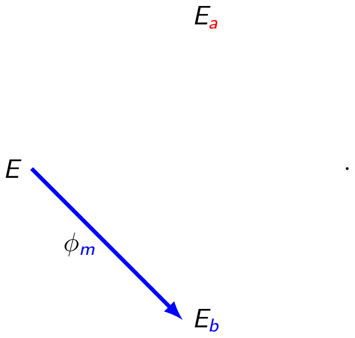
- *Bob* computes an isogeny $\phi_m : E \mapsto E/\langle P_m + bQ_m \rangle$ from his secret exponent b and sends $\phi_m(P_\ell), \phi_m(Q_\ell), E_b = E/\langle P_m + bQ_m \rangle$;



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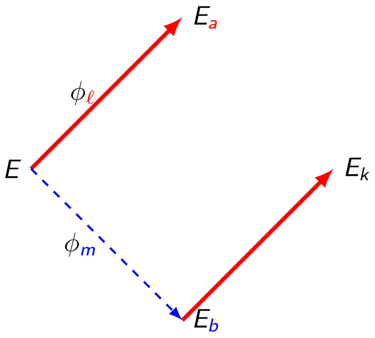

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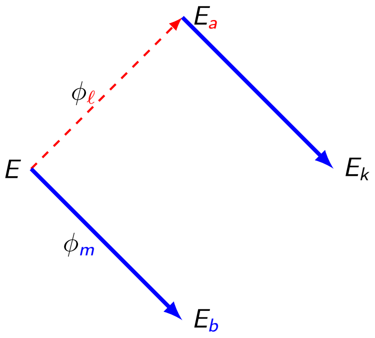

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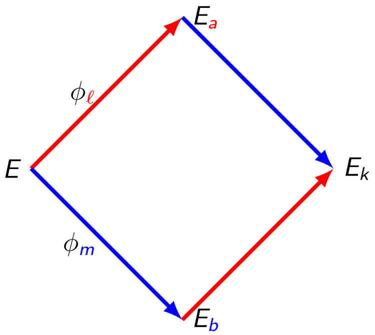
  Bob recepts $E_a, \phi_\ell(P_m), \phi_\ell(Q_m)$ from Alice and computes $k = j(E_a/\langle \phi_\ell(P_m) + b\phi_\ell(Q_m) \rangle))$.
- They share a common secret key $k = j(E/\langle P_m + bQ_m \rangle, \langle P_\ell + aQ_\ell \rangle) = j(E_k)$



Computation of the public parameters

We follow the example of SikeP503 of [JAC+17]

```
? e_2=250; e_3=159; p=2^e_2*3^e_3-1;
? isprime(p)
% = 1
```

We first define the finite field $\mathbb{F}_{p^2}$ as follows

```
? P=(X^2+1)*Mod(1,p);
? a=ffgen(P,'a);
```

Now we define a curve on $\mathbb{F}_{p^2}$ that we know is supersingular :

```
? E=ellinit([0,1],a);
```

We can check that this curve has the good structure :

```
? E.cyc
% = [1317584315690711738083925 [...] 961154237431808,
 1317584315690711738083925 [...] 961154237431808]
```

Here we just saw the structure of the group of points of E. We now facotrize this result for some clarity :

```
? factor(%[1])
% =
[2 250]

[3 159]
```

Now we will determine a basis of $E[2^{250}] \simeq \mathbb{Z}/2^{250}\mathbb{Z} \times \mathbb{Z}/2^{250}\mathbb{Z}$

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We look for a candidate point by incrementing the possible abscissae :

```
? x_0=Mod(1,p); issquare(x_0^3+x_0)
% = 1
```

The first one is indeed the abscissa of a point of the elliptic curve, we compute then the ordinate and the order of this point.

```
? issquare(x_0^3+x_0,&y_0);
//or y_0=ellordinate(E,x_0)[2];
? ellisoncurve(E,[x_0,y_0])
% = 1
? P=[x_0,y_0]; factor(ellorder(E,P))
%10 =
[2 247]

[3 156]
```

We see that this point is not of order high enough, thus we search for another point.

```
? x_0=Mod(14,p); issquare(x_0^3+x_0,&y_0)
= 1
```

To obtain directly a point of order 2^{250} we multiply this point directly by the cofactor of the order of the group of points of E : $3^{159} = 3^{e_3}$.

```
? P_2=ellmul(E0,[x_0,y_0],3^(e_3));
? factor(ellorder(E0,P_2))
% =
[2 250]
```

This point P_2 has good order.

Now we search for a second point to form a basis of $E[2^{250}]$.

```
? x_0=a+4; issquare(x_0^3+x_0)
% = 1
? issquare(x_0^3+x_0,&y_0); \
Q_2=ellmul(E,[x_0,y_0],3^(e_3)); factor(ellorder(E,Q_2))
% =
[2 250]
```

We have found a second point Q_2 of order 2^{250}, we have then to test if it forms a basis of $E[2^{250}]$ with the Weil pairing :

```
? t= ellweilpairing(E,P_2,Q_2,2^(e_2))
% = 414[...]58013*a + 1966154683551[...]38443
? t^{2^(e_2-1)}
% = 1317584315690711738083 9[...]1154237431806
```

Since the Weil pairing is of maximal order the points P_2, Q_2 do form a basis of $E[2^{250}]$.

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Now we can compute the secret isogeny ϕ_2 generated from the secret sk_2,

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Now we can compute the secret isogeny ϕ_2 generated from the secret sk_2, but since it is an isogeny of great order we won't compute it directly. We will compute iteratively the 250 2-isogeny as follows.

```
? sk2=random(2^{e_2}); R_2=ellmul(E,Q_2,sk2);
? S_2=elladd(E,P_2,R_2); S_int=ellmul(E,S_2,2^(e_2-1));
? iso=ellisogeny(E,S_int)
 = [[0, 0, 0, 13175843156[...]1154237431803, 0],
  [x^3 + x, y*x^3 +
  131758431569071173808392[...]019593961154237431806*y*x,
   x]]
```

A more clever and optimized method is proposed in [LDJ14, JAC+17]. The first output of ellisogeny is the codomain curve and the second output is the "mapping" itself.

```
? Eb=ellinit(iso[1]);
```

Once we have defined a basis of $E[3^{159}]$:

```
? Q_3x=0x1e7d6 [...] d7e39b6997f70023e0a23b4b3787ef08f;
? Q_3y=0x2ec0a [...] c8ad47064f05c06dc5d4aae61ccceff1f26;
? Q_3=[Q_3x,Q_3y];
? P_3x1=0x21b7 [...] a67dd7ed98b9793685fa2e22d6d89d66a4e;
? P_3x2=0x2f37 [...]9ceb53821d3e8012f7f391f57364f402909;
? P_3y1=0x78f8 [...] efee6010cdf34a7de9f9e239b103e7b3eee;
? P_3y2=0x37f3 [...]61d04f9f3a8317f7916e016f2733b828ac0;
? P_3x=P_3x1+a*P_3x2;  P_3y=P_3y1+a*P_3y2;P_3=[P_3x,P_3y]
```

One can check that the isogeny preserves the order of points of $E[3^{159}]$

```
? factor(ellorder(Eb,ellisogenyapply(iso[2],P_3)))
% =
[3 159]

? factor(ellorder(Eb,ellisogenyapply(iso[2],Q_3)))
% =
[3 159]
```

We generalize those computations of isogenies with the following similar functions :

- *isogen*$_\ell$
- *isoex*$_\ell$

They are viewable in `definition_fonctions.gp` (gp2). They only consist in the following algorithm :

isogen$_\ell$

Entrée: Bases of $E[\ell^{e_\ell}] = \langle P_\ell, Q_\ell \rangle$ and $E[m^{e_m}] = \langle P_m, Q_m \rangle$ and a secret exponent sk_ℓ

Sortie: The image of the basis $E[m^{e_m}]$ by the isogeny specified by the secret exponent sk_ℓ and $E[\ell^{e_\ell}]$

Compute $S_\ell = P_\ell + [sk_\ell]Q_\ell$

for $i = 1$ to e_ℓ **do**

  Compute $\phi : E \mapsto E/\langle S_\ell \rangle = E_b$

  Compute $(P_m, Q_m, S_\ell, E) \leftarrow (\phi(P_m), \phi(Q_m), [\ell^{e_\ell - i - 1}]\phi(S_\ell)), E_b)$

end for

return (P_m, Q_m, E)

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Computing a common secret key

Thus using those functions and the points we have computed we can share a common secret key :

```
? R_2 = ellsub(E, P_2, Q_2); //not useful here
? R_3 = ellsub(E, P_3, Q_2); //not useful here
? sk3=random(3^{e_3});
? pk2 = isogen_ell(E, P_2, Q_2, sk2, P_3,
 Q_3, R_3, e_2, 2);
? pk3 = isogen_ell(E, P_3, Q_3, sk3, P_2,
Q_2, R_2, e_3, 3);
```

```
? j_ech1 = isoex_ell( pk2[4], sk3, pk2[1],
  pk2[2], pk2[3], e_3, 3)
% = 95211232169147321 37[...]0222351872676569 04887*a +
  6307916662[...]24346811520916299511477286097663421965
? j_ech2 = isoex_ell( pk3[4], sk2, pk3[1],
pk3[2], pk3[3], e_2, 2)
% = 95211232169147321 37[...]0222351872676569 04887*a +
6307916662[...]24346811520916299511477286097663421965
? j_ech1==j_ech2
% = 1
```

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```
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 pk2[2], pk2[3], e_3, 3)
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? j_ech2 = isoex_ell( pk3[4], sk2, pk3[1],
pk3[2], pk3[3], e_2, 2)
% = 9521123216914732137[...]022235187267656904887*a +
6307916662[...]24346811520916299511477286097663421965
? j_ech1==j_ech2
% = 1
```

In the end we get the same *j*-invariant.

Thanks for your attention.

Whitfield Diffie and Martin E. Hellman.
New directions in cryptography.
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