Explicit small image theorem for residual modular representations

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January 22th, 2020

Notations

$G_{\mathbb{Q}}$	Absolute Galois group of $\mathbb{Q}$
$f = \sum_{n=1}^{\infty} a_n(f) q^n \in S_k^{\mathrm{new}}(N, \varepsilon)$	parabolic, normalized, new eigenform of weight k, level N and character ε
K_f	coefficient field of f
λ	place of K_f above a prime number ℓ
$\mathbb{F}_\lambda$	residual field of λ

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Deligne attached in 1972 to f and λ, a semi-simple residual Galois representation

$$\overline{\rho}_{f,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\lambda),$$

unramified outside $N\ell$ and such that

$$\mathrm{Tr}(\overline{\rho}_{f,\lambda}(\mathrm{Frob}_p)) = a_p(f), \text{for } p \nmid N\ell$$

$$\det(\overline{\rho}_{f,\lambda}) = \overline{\chi}_\ell^{k-1}\varepsilon$$

Ribet's theorem

Theorem (Ribet, 1985)

- For all but finitely many places λ, $\overline{\rho}_{f,\lambda}$ is irreducible;
- If f is not CM, for all but finitely many places λ, ℓ divides $|\overline{\rho}_{f,\lambda}(G_{\mathbb{Q}})|$.

This finite set of places is called exceptional places.

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$\Delta \in \mathrm{S}_{12}(1)$: $\overline{\rho}_{\Delta,\ell}$ is reducible for $\ell = 2, 3, 5, 7, 691$ and irreducible but of order prime to ℓ for $\ell = 23$.

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2. Can we explicitly compute the "exceptional" places?

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- If $\overline{\rho}_{f,\lambda}$ has dihedral projective image, then $\ell \leqslant (2(8kN^2(2\log\log(N)+2.4))^{\frac{k-1}{2}})^{[K_f:\mathbb{Q}]}$;

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If $\overline{\rho}_{f,\lambda}$ is reducible, then one of the following applies:

1. $\ell \mid N\phi(N)$, or $\ell \leqslant k+1$;
2. There exists two primitive Dirichlet characters ε_1, ε_2 of conductor $\mathfrak{c}_1$, $\mathfrak{c}_2$ such that $\varepsilon_1\varepsilon_2 = \varepsilon$, $\mathfrak{c}_1\mathfrak{c}_2 \mid N$ and ℓ divides one of the following:
   - The norm of $p^k - \varepsilon_1(p)\overline{\varepsilon_2}(p)$ for $p \mid N$;
   - The norm of $p^k - (\varepsilon_1\overline{\varepsilon_2})_0(p)$ for $p \mid N$;
   - The numerator of the norm of $\frac{B_{k,(\varepsilon_1\overline{\varepsilon_2})_0}}{2k}$,

where $\sum_{k=0}^{\infty} \frac{B_{k,\chi}}{k!} t^k = \sum_{n=1}^{\mathfrak{c}-1} \chi(n) \frac{te^{nt}}{e^{\mathfrak{c}t}-1}$, and χ_0 is the primitive character associated to χ.

Reducible case: general idea

If $\overline{\rho}_{f,\lambda}$ is reducible then

$$\overline{\rho}_{f,\lambda}^{\mathrm{ss}} \cong \overline{\chi}_\ell^a \nu_1 \oplus \overline{\chi}_\ell^b \nu_2,$$

with $\nu_i : (\mathbb{Z}/\mathfrak{c}_i\mathbb{Z})^\times \to \overline{\mathbb{F}_\ell}^\times$ primitive, $0 \leqslant a \leqslant b \leqslant \ell - 2$.

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$\overline{\chi}_\ell^{k-1}\varepsilon \equiv \overline{\chi}_\ell^{a+b}\nu_1\nu_2 \pmod{\lambda}$;
$a_p(f) \equiv p^a\nu_1(p) + p^b\nu_2(p) \pmod{\lambda}$ for $p \nmid N\ell$.

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Moreover $\mathfrak{c}_1\mathfrak{c}_2 \mid N$.

Assuming $\ell \nmid N\phi(N)$ and $\ell > k+1$, we get $a = 0, b \equiv k-1 \pmod{\ell - 1}$, $\varepsilon_1\varepsilon_2 = \varepsilon$ (with ε_i the multiplicative lift of ν_i).

Eisenstein series

For $k \geqslant 2$, ε_i primitive Dirichlet character mod $\mathfrak{c}_i$.

$$E = C + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \varepsilon_1 \left(\frac{n}{d} \right) \varepsilon_2(d) \right) q^n \in \mathrm{M}_k(\mathfrak{c}_1\mathfrak{c}_2, \varepsilon_1\varepsilon_2),$$

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We get for all $p \nmid N\ell$, $a_p(f) \equiv a_p(E) \pmod{\lambda}$.

Reducible case: general idea II

Proposition

Let $\pi_M(f) := \sum_{(n,M)=1} a_n(f) q^n$. It's a modular form of weight k and level $N \cdot \prod_{p|M} p \prod_{p|M, p\nmid N} p$.

In particular $\pi_{N\ell}(f) \in \begin{cases} \mathrm{M}_k(N\ell \operatorname{rad}(N)) & \text{if } \ell \mid N \\ \mathrm{M}_k(N\ell^2 \operatorname{rad}(N)) & \text{if } \ell \nmid N \end{cases}$.

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Explicit computations of the constant term of $\pi_{N\ell}(E)$ at the cusps give the result.

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3. Factor those norms in prime factors;
4. The possible reducible primes are these factors together with the primes $\ell \leqslant k+1$ or $\mid N\phi(N)$.

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with $\chi : (\mathbb{Z}/17\mathbb{Z})^\times \to \mathbb{C}^\times$ sending 3 to $e^{\frac{2i\pi}{16}}$.

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Dihedral case:

- Astronomical bound.

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So for $p \nmid N\ell$, $a_p(f) \equiv p^a a_p(E) \pmod{\lambda}$ for a well chosen Eisenstein series.

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$$E = \begin{cases} E^{\varepsilon_1,\varepsilon_2}_{b-a+1} & \text{if } b-a>1 \text{ and } (b-a,\varepsilon_1,\varepsilon_2) \neq (1,1,1) \\ E^{\varepsilon_1,\varepsilon_2}_{b-a+\ell} & \text{if } \begin{cases} (b-a,\varepsilon_1,\varepsilon_2) = (1,1,1) \\ \text{or } b-a=0 \text{ and } (\ell,\varepsilon_1,\varepsilon_2) \neq (2,1,1) \end{cases} \\ E_4 & \text{if } (b=a, \ell=2, \varepsilon_1=\varepsilon_2=1) \end{cases}$$

Reducible case: explicit check II

The following are equivalent:

1. $\overline{\rho}^{\mathrm{ss}}_{f,\lambda} \cong \overline{\chi}_\ell^a \overline{\varepsilon_1} \oplus \overline{\chi}_\ell^b \overline{\varepsilon_2}$ as before;
2. for all primes $p \nmid N\ell$, $a_p(f) \equiv p^a a_p(E) \pmod{\lambda}$;

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2. for all primes $p \nmid N\ell$, $a_p(f) \equiv p^a a_p(E) \pmod{\lambda}$;
3. $\pi_{N\ell}(f) \equiv \mathcal{E} \pmod{\lambda}$ with

$$\mathcal{E} \in \mathrm{M}_{k_E + a(\ell+1)}(N') \text{ such that } \mathcal{E} \equiv \sum_{(n,N\ell)=1} n^a a_n(E) q^n \pmod{\lambda};$$

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4. for all primes $p \leqslant \dfrac{\max(k, k_E + a(\ell+1))N}{12} \prod_{q|N\ell}(q+1)$, $p \nmid N\ell$,

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Proposition

If $\overline{\rho}_{f,\lambda}$ is reducible for $\ell > k+1$, $\ell \nmid N\phi(N)$, then

$$\pi_{N\ell}(f) \equiv \pi_{N\ell}(E_k^{\varepsilon_1,\varepsilon_2}) \pmod{\lambda},$$

with $\varepsilon_1 \varepsilon_2 = \varepsilon$.

Algorithm

1. For all $\varepsilon_1\varepsilon_2 = \varepsilon$, compute $\gcd_{p \leqslant B, p \nmid N} p \cdot \left(a_p(f) - a_p(E_k^{\varepsilon_1,\varepsilon_2})\right)$, these are the reducible primes bigger than $k+1$ and not dividing $N\phi(N)$;

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2. For all $\ell \leqslant k+1$ or $\ell \mid N\phi(N)$, compute the possible Dirichlet characters and powers of the cyclotomic character of the decomposition;

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2. For all $\ell \leqslant k+1$ or $\ell \mid N\phi(N)$, compute the possible Dirichlet characters and powers of the cyclotomic character of the decomposition;
3. For all set of parameters, check the previous congruences.

Thank you for your attention!