Parametrizing $\mathbb{Q}$-curves by modular units

François Brunault
(joint work with Hang Liu and Haixu Wang)

January 12, 2022

Atelier PARI/GP, Besançon

Motivation: Mahler measures and L-functions

Mahler measures

Definition

The Mahler measure of a Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is

$$m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| dt_1 \cdots dt_n.$$

- For $P \in \mathbb{C}[x]$ monic, Jensen's formula gives $m(P) = \sum_{\substack{P(\alpha)=0 \\ |\alpha| \geq 1}} \log |\alpha|$.
- If P has coefficients in $\overline{\mathbb{Q}}$, then $m(P)$ is a period in the sense of Kontsevich and Zagier.
- In favorable situations, $m(P)$ is (often conjecturally) related to L-functions. For example (Smyth, 1981):

$$m(1 + x + y) = L'(\chi_{-3}, -1)$$
$$m(1 + x + y + z) = -14\zeta'(-2).$$

Mahler measures

Boyd and Deninger discovered experimentally in 1997:

$$m\Big(x + \frac{1}{x} + y + \frac{1}{y} + 1\Big) \stackrel{?}{=} L'(E,0) = \frac{15}{4\pi^2} L(E,2)$$

where E is the elliptic curve with affine equation $x + \frac{1}{x} + y + \frac{1}{y} + 1 = 0$.

Boyd also found *families* of identities, for example

$$m\Big(x + \frac{1}{x} + y + \frac{1}{y} + k\Big) \stackrel{?}{=} r_k \cdot L'(E_k,0) \qquad (k \in \mathbb{Z}\backslash\{0,\pm 4\},\ r_k \in \mathbb{Q}^\times).$$

Only finitely many identities are proven: $|k| \in \{1,2,3,5,8,12,16\}$.

The proof requires E_k to be parametrized by *modular units*.

More precisely, we need $\varphi \colon X_1(N_k) \to E_k$ such that $\varphi^*(x)$ and $\varphi^*(y)$ are modular units. Outline of the proof:

$$m(P_k) \stackrel{\text{Jensen}}{=} \int_\gamma \eta(x,y) = \int_{\widetilde{\gamma}} \eta(\varphi^*(x),\varphi^*(y)) \stackrel{\substack{\text{Rogers-}\\ \text{Zudilin}}}{=} L'(f_k,0).$$

Mahler measures

Objectives

- Discover new identities for Mahler measures of genus 1 polynomials.
- Prove them in a systematic way (when modular units are available).
- Determine whether an elliptic curve admits a parametrization by modular units.
- Generalize to elliptic curves over number fields and higher genus curves which are parametrized by modular curves.

Specifically, we will consider $\mathbb{Q}$-*curves*.

$\mathbb{Q}$-curves

$\mathbb{Q}$-curves

Definition

A $\mathbb{Q}$-curve is an elliptic curve defined over $\overline{\mathbb{Q}}$ which is isogenous to all its Galois conjugates.

Example

Let K be a real quadratic field, and $u \in K \backslash \{\pm 1\}$ such that $4u \in \mathcal{O}_K$ and $\mathrm{N}_{K/\mathbb{Q}}(u) = 1$. Then $E_k \colon x + \frac{1}{x} + y + \frac{1}{y} + 4u = 0$ is a $\mathbb{Q}$-curve.

In this case the isogeny is defined over K.

Modularity theorem (Khare–Wintenberger, Ribet)

Let E be an elliptic curve over $\overline{\mathbb{Q}}$. Then E is a $\mathbb{Q}$-curve if and only if there exists a modular parametrization $\varphi \colon X_1(N)_{\overline{\mathbb{Q}}} \to E$.

Question. Can we make φ explicit?

$\mathbb{Q}$-curves and modular forms

Let $\varphi\colon X_1(N)_{\overline{\mathbb{Q}}} \to E$ be a modular parametrization.

Then $\varphi^*(\omega_E) = \omega_f = 2\pi i f(\tau)d\tau$ for some $f \in S_2(\Gamma_1(N))$ (not necessarily a newform!). Moreover

$$\Lambda_f := \Big\{ \int_\gamma \omega_f : \gamma \in H_1(X_1(N), \mathbb{Z}) \Big\}$$

is a lattice in $\mathbb{C}$, and we have $E(\mathbb{C}) \cong \mathbb{C}/\Lambda_f$.

Conversely, let $f \in S_2(\Gamma_1(N))$ such that Λ_f is a lattice in $\mathbb{C}$. Then $E_f = \mathbb{C}/\Lambda_f$ is a $\mathbb{Q}$-curve with modular parametrization

$$\varphi\colon X_1(N)_{\overline{\mathbb{Q}}} \to E_f, \quad \tau \mapsto \Big[\int_0^\tau \omega_f \Big].$$

Questions. Given E, can we compute f, and conversely? Can we compute φ? (and what does this mean?)

Computing the modular parametrization

Overview

Input: a modular form $f \in S_2(\Gamma_1(N))$ such that Λ_f is a lattice in $\mathbb{C}$.

We always assume $f = \sum_\sigma c_\sigma F^\sigma$ is a $\overline{\mathbb{Q}}$-linear combination of the Galois conjugates F^σ of a newform F in $S_2(\Gamma_1(N))$.

Goals:

- Compute the $\mathbb{Q}$-curve E_f in Weierstrass form.
- Determine if E_f can be parametrized by modular units.
- If so, compute φ in algebraic form. By this we mean finding two modular units $u, v \in \overline{\mathbb{Q}}(X_1(N))$ such that $\overline{\mathbb{Q}}(E_f) \cong \overline{\mathbb{Q}}(u, v)$.

We will construct u and v using Siegel units

$$g_{a,b}(\tau) = q^\alpha \prod_{\substack{n \geq 0 \\ n \equiv a \bmod N}} (1 - q^{n/N}\zeta_N^b) \prod_{\substack{n \geq 1 \\ n \equiv -a \bmod N}} (1 - q^{n/N}\zeta_N^{-b}).$$

where $a, b \in \mathbb{Z}/N\mathbb{Z}$, $\alpha = B_2(\{a/N\})$, $q^\alpha = e^{2\pi i \alpha \tau}$, $\zeta_N = e^{2\pi i/N}$.

Step 1: The lattice Λ_f

Recall that $E_f = \mathbb{C}/\Lambda_f$ with $\Lambda_f = \{\int_\gamma \omega_f : \gamma \in H_1(X_1(N), \mathbb{Z})\}$. The map

$$\Gamma_1(N) \to H_1(X_1(N), \mathbb{Z}), \quad g \mapsto \{0, g0\}$$

is a surjective group morphism.

1. Compute generators $g_1, \ldots, g_r$ of $\Gamma_1(N)$ (more generally, $\Gamma_H(N)$) using `msfarey` and `mspolygon`.
2. For each $1 \leq i \leq r$, compute $I(g_i) = \int_0^{g_i 0} \omega_f$ using `mfsymboleval`.
3. Compute $\mathbb{Z}$-generators of $\Lambda_f = \langle I(g_1), \ldots, I(g_r) \rangle$ using `lindep` and `qflll`.

Step 2: The elliptic curve E_f

The elliptic curve $E_f = \mathbb{C}/\Lambda_f$ has Weierstrass equation

$$E_f : y^2 = x^3 - 27c_4(\Lambda_f)x - 54c_6(\Lambda_f).$$

Hypothesis: $c_4(\Lambda_f), c_6(\Lambda_f) \in \mathbb{Q}(\zeta_N)$.

(This does not always hold.)

1. Compute c_4, c_6 as complex numbers.
2. Reconstruct c_4, c_6 in $\mathbb{Q}(\zeta_N)$ using `lindep`.

We will see later how to check the Weierstrass equation is correct.

Step 3: Images of cusps

Recall that $\varphi\colon X_1(N) \to E_f$ is given by $\tau \mapsto [\int_0^\tau \omega_f]$.

Hypothesis: φ is defined over $\mathbb{Q}(\zeta_N)$.

(This does not always hold.)

1. Enumerate the cusps $c_1, \ldots, c_s$ of $X_1(N)$.
2. For each $1 \leq i \leq s$, compute $z_i = \int_0^{c_i} \omega_f$.
3. Compute $p_i = \texttt{ellztopoint}(E_f, z_i) \in E_f(\mathbb{C})$.
4. Writing $p_i = (x_i, y_i)$, reconstruct x_i, y_i in $\mathbb{Q}(\zeta_N)$ using `lindep`.
5. Check whether $p_i \in E_f(\mathbb{Q}(\zeta_N))$.

Step 4: Admissible points

We want to find functions on E_f whose pull-back to $X_1(N)$ are modular units. We define

$$S = \{p \in E_f : \varphi^{-1}(p) \subset \{\mathrm{cusps}\}\} \subset \{p_1, \ldots, p_s\}.$$

Then for any function h on E supported in S, $\varphi^*(h)$ is a modular unit.

1. Compute the modular degree $\deg(\varphi)$ using `mfpetersson` and
$$\int_{X_1(N)} \omega_f \wedge \overline{\omega_f} = \deg(\varphi) \cdot \int_{E_f} \omega_{E_f} \wedge \overline{\omega_{E_f}}.$$
2. For each cusp c, compute the ramification index $e_\varphi(c)$ using `mfslashexpansion`.
3. For each point $p \in \varphi(\{\mathrm{cusps}\})$, check whether
$$\sum_{\substack{c \text{ cusp} \\ \varphi(c)=p}} e_\varphi(c) = \deg(\varphi).$$

   If true, put p in S.

Step 5: The function field of E_f

We want to find two functions h_1, h_2 on E_f whose zeros and poles are contained in S, and which generate the function field of E_f.

If $|S| \leq 2$, this is impossible.

If $|S| \geq 3$:

1. Generate principal divisors on E supported in S (this is possible since S consists of torsion points, by the Manin-Drinfeld theorem).
2. Take two such divisors D_1, D_2 and compute functions $h_1, h_2 \in \overline{\mathbb{Q}}(E)$ having these divisors.
3. Compute the minimal polynomial $P \in \overline{\mathbb{Q}}[X_1, X_2]$ of (h_1, h_2).
4. Check the partial degrees of P to decide whether $\overline{\mathbb{Q}}(E_f) = \overline{\mathbb{Q}}(h_1, h_2)$.

If h_1, h_2 satisfy this condition, then $P(X_1, X_2) = 0$ is a model of E_f.

Step 6: Certifying the parametrization

Because our computations were numerical, we haven't proved the parametrization exists yet!

1. Compute the q-expansion of $\varphi^*(x)$ and $\varphi^*(y)$ in $\mathbb{Q}(\zeta_N)((q))$.

$$\begin{cases} x = u^2 q^{-2e} + O(q^{-2e+1}) \\ y = u^3 q^{-3e} + O(q^{-3e+1}) \end{cases}$$

   with $e = e_\varphi(\infty)$ and $u \in \mathbb{Q}(\zeta_N)^\times$, exactly as in `elltaniyama`: use the two equations $y^2 = x^3 - 27c_4 x - 54c_6$ and $\omega_f = dx/2y$ to determine inductively the Fourier coefficients of x and y.

2. Deduce the q-expansions of h_1 and h_2.
3. Express h_1, h_2 as products of Siegel units by comparing the divisors and checking the leading coefficient.

Step 6: Certifying the parametrization

Each h_i is of the form

$$C \prod_{a,b \in \mathbb{Z}/N\mathbb{Z}} g_{a,b}^{e_{a,b}} \qquad (C \in \mathbb{Q}(\zeta_N)^\times, e_{a,b} \in \mathbb{Z}).$$

4. Prove that these products are indeed modular for $\Gamma_1(N)$ (in general, such a product is only modular for $\Gamma(12N^2)$). This uses a criterion of Kubert–Lang.
5. Denoting by u_1, u_2 these modular units, prove that $P(u_1, u_2) = 0$ by checking the q-expansion to high enough accuracy.

The data (P, u_1, u_2) *certifies* the modular parametrization.

We can also certify the images of the cusps computed previously.

Question. How to describe and certify a modular parametrization when no modular unit is available?

Examples

Thank you!