Zeta function

A tutorial

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Riemann ζ function

```
? zeta(2)
%1 = 1.6449340668482264364724151666460251892
? Pi^2/6
%2 = 1.6449340668482264364724151666460251892
? zeta(-11)
%3 = 0.021092796092796092796092796092796092796
? bestappr(%)
%4 = 691/32760
? -bernfrac(12)/12
%5 = 691/32760
? zeta(1+x+O(x^5))
%6 = 1.0*x^-1+0.577215665+0.0728158455*x-0.00484518
? Euler
%7 = 0.57721566490153286060651209008240243104
```

Γ function

```
? gamma(1/2)
%8 = 1.772453850905516027298167483341451828
? sqrt(Pi)
%9 = 1.772453850905516027298167483341451828
? gamma'(1)
%10 = -0.57721566490153286060651209008240243105
? gamma(s)*gamma(1-s)
%11 = s^-1+1.64493407*s+1.89406566*s^3+1.97110218*s
? Pi/sin(Pi*s)
%12 = 1.*s^-1+1.64493407*s+1.89406566*s^3+1.9711021
```

Λ function

```
? lambda(s)=zeta(s)*Pi^-(s/2)*gamma(s/2);
? lambda(I)
%14 = -0.47715243067693608 9+0.49975517381751505 2*I
? lambda(1-I)
%15 = -0.47715243067693608 9+0.49975517381751505 2*I
? lfunlambda(1,I)
%16 = -0.47715243067693608 91+0.49975517381751505 21*
```

multizeta

```
? zetamult([2,1])
%17 = 1.2020569031595942853997381615114499908
? zeta(3)
%18 = 1.2020569031595942853997381615114499908
? zetamult([3,5])+zetamult([5,3])+zeta(8)
%19 = 1.2464461661478693187224801387561935590
? zeta(3)*zeta(5)
%20 = 1.2464461661478693187224801387561935590
? a = zetamult([2,2,2])
%21 = 0.19075182412208421369647211183579759898
? b = zetamult([3,3])
%22 = 0.21379886822459254709958357450803364964
? bestappr(zeta(6)/a)
%23 = 16/3
? bestappr(zeta(6)/b)
```

multizeta

```
? c = zetamult([3,2,1])
%25 = 0.032309028991669881698406491680195415633
? lindep([a,b,c])
%26 = [59,-54,9]~
? 59*a-54*b+9*c
%27 = 1.9101783200862172004E-38
? V=zetamultall(6);
? V[zetamultconvert([3,2,1],2)]
%29 = 0.032309028991669881698406491680195415633
```

polylogarithms

```
? dilog(1)
%30 = 1.6449340668482264364724151666460251892
? dilog(I)
%31 = -0.20561675835602830444+0.91596559417721901505*
? -Pi^2/48+I*Catalan
%32 = -0.20561675835602830466+0.91596559417721901505*
? polylog(3,1)
%33 = 1.2020569031595942853997381615114499908
```

Morita Γ function

Gross-Koblitz formula:

```
? gamma(1/4 + O(5^10))
%34 = 1 + 4*5 + 3*5^4 + 5^6 + 5^7 + 4*5^9 + O(5^10)
? algdep(%,4)
%35 = x^4 + 4*x^2 + 5
```

Kubota-Leopoldt ζ function

```
? zeta(-3+O(5^20))
%36 = 4*5^-1+4+3*5+4*5^3+4*5^5+4*5^7+4*5^9+4*5^11+4
? bestappr(%)
%37 = -31/30
? (5^3-1)*bernfrac(4)/4
%38 = -31/30
```

Hurwitz ζ function

$$\zeta(s,x) = \sum_{n>=0} (n+x)^{-s}$$

```
? real(zetahurwitz(2,I))
%39 = -0.5369999033772362137016734818
? (-Pi^2/sinh(Pi)^2-1)/2
%40 = -0.5369999033772362137016734818
? zetahurwitz(0,1,1)
%41 = -0.91893853320467274178032973640561763986
? -log(2*Pi)/2
%42 = -0.91893853320467274178032973640561763986
? zetahurwitz(-3+O(5^20),1)
%43 = 4*5^-1+4+3*5+4*5^3+4*5^5+4*5^7+4*5^9+4*5^11+4
```

Help

```
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```

Help

```
? ?4
? ?atan

atan(x): arc tangent of x.


? ??atan

atan(x):

    Principal branch of tan^{-1}(x)  = log ((1+ix)/(

    The library syntax is GEN gatan(GEN x, long prec
```