User's Guide

to

the PARI library

(version 2.15.4)

The PARI Group

Institut de Mathématiques de Bordeaux, UMR 5251 du CNRS.
Université de Bordeaux, 351 Cours de la Libération
F-33405 TALENCE Cedex, FRANCE
`e-mail: pari@math.u-bordeaux.fr`

Home Page:
`http://pari.math.u-bordeaux.fr/`

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Chapter 4:
Programming PARI in Library Mode

The *User's Guide to Pari/GP* gives in three chapters a general presentation of the system, of the `gp` calculator, and detailed explanation of high level PARI routines available through the calculator. The present manual assumes general familiarity with the contents of these chapters and the basics of ANSI C programming, and focuses on the usage of the PARI library. In this chapter, we introduce the general concepts of PARI programming and describe useful general purpose functions; the following chapters describes all public low or high-level functions, underlying or extending the GP functions seen in Chapter 3 of the User's guide.

4.1 Introduction: initializations, universal objects.

To use PARI in library mode, you must write a C program and link it to the PARI library. See the installation guide or the Appendix to the *User's Guide to Pari/GP* on how to create and install the library and include files. A sample Makefile is presented in Appendix A, and a more elaborate one in `examples/Makefile`. The best way to understand how programming is done is to work through a complete example. We will write such a program in Section 4.10. Before doing this, a few explanations are in order.

First, one must explain to the outside world what kind of objects and routines we are going to use. This is done* with the directive

```
#include <pari/pari.h>
```

In particular, this defines the fundamental type for all PARI objects: the type **GEN**, which is simply a pointer to `long`.

Before any PARI routine is called, one must initialize the system, and in particular the PARI stack which is both a scratchboard and a repository for computed objects. This is done with a call to the function

```
void pari_init(size_t size, ulong maxprime)
```

The first argument is the number of bytes given to PARI to work with, and the second is the upper limit on a precomputed prime number table; `size` should not reasonably be taken below 500000 but you may set `maxprime` $= 0$, although the system still needs to precompute all primes up to about 2^{16}. For lower-level variants allowing finer control, e.g. preventing PARI from installing its own error or signal handlers, see Section 5.1.2.

We have now at our disposal:

• a PARI *stack* containing nothing. This is a big connected chunk of `size` bytes of memory, where all computations take place. In large computations, intermediate results quickly clutter up memory so some kind of garbage collecting is needed. Most systems do garbage collecting when the memory is getting scarce, and this slows down the performance. PARI takes a different approach,

* This assumes that PARI headers are installed in a directory which belongs to your compiler's search path for header files. You might need to add flags like `-I/usr/local/include` or modify `C_INCLUDE_PATH`.

admittedly more demanding on the programmer: you must do your own cleaning up when the intermediate results are not needed anymore. We will see later how (and when) this is done.

• the following *universal objects* (by definition, objects which do not belong to the stack): the integers 0, 1, −1, 2 and −2 (respectively called `gen_0`, `gen_1`, `gen_m1`, `gen_2` and `gen_m2`), the fraction $\frac{1}{2}$ (`ghalf`). All of these are of type `GEN`.

• a *heap* which is just a linked list of permanent universal objects. For now, it contains exactly the ones listed above. You will probably very rarely use the heap yourself; and if so, only as a collection of copies of objects taken from the stack (called clones in the sequel). Thus you need not bother with its internal structure, which may change as PARI evolves. Some complex PARI functions create clones for special garbage collecting purposes, usually destroying them when returning.

• a table of primes (in fact of *differences* between consecutive primes), called **diffptr**, of type `byteptr` (pointer to `unsigned char`). Its use is described in Section 5.4 later. Using it directly is deprecated, high-level iterators provide a cleaner and more flexible interface, see Section 4.8.2 (such iterators use the private prime table, but extend it dynamically).

• access to all the built-in functions of the PARI library. These are declared to the outside world when you include `pari.h`, but need the above things to function properly. So if you forget the call to `pari_init`, you will get a fatal error when running your program.

4.2 Important technical notes.

4.2.1 Backward compatibility. The PARI function names evolved over time, and deprecated functions are eventually deleted. The file `pariold.h` contains macros implementing a weak form of backward compatibility. In particular, whenever the name of a documented function changes, a `#define` is added to this file so that the old name expands to the new one (provided the prototype didn't change also).

This file is included by `pari.h`, but a large section is commented out by default. Define `PARI_OLD_NAMES` before including `pari.h` to pollute your namespace with lots of obsolete names like `un`*: that might enable you to compile old programs without having to modify them. The preferred way to do that is to add `-DPARI_OLD_NAMES` to your compiler `CFLAGS`, so that you don't need to modify the program files themselves.

Of course, it's better to fix the program if you can!

4.2.2 Types.

Although PARI objects all have the C type `GEN`, we will freely use the word **type** to refer to PARI dynamic subtypes: `t_INT`, `t_REAL`, etc. The declaration

```
GEN x;
```

declares a C variable of type `GEN`, but its "value" will be said to have type `t_INT`, `t_REAL`, etc. The meaning should always be clear from the context.

* For `(long)gen_1`. Since 2004 and version 2.2.9, typecasts are completely unnecessary in PARI programs.

4.2.3 Type recursivity.

Conceptually, most PARI types are recursive. But the `GEN` type is a pointer to `long`, not to `GEN`. So special macros must be used to access `GEN`'s components. The simplest one is `gel`(V, i), where **el** stands for **el**ement, to access component number i of the `GEN` V. This is a valid `lvalue` (may be put on the left side of an assignment), and the following two constructions are exceedingly frequent

```
gel(V, i) = x;
x = gel(V, i);
```

where `x` and `V` are `GEN`s. This macro accesses and modifies directly the components of V and do not create a copy of the coefficient, contrary to all the library *functions*.

More generally, to retrieve the values of elements of lists of ... of lists of vectors we have the `gmael` macros (for **m**ultidimensional **a**rray **el**ement). The syntax is **gmael**$n(V, a_1, \ldots, a_n)$, where V is a `GEN`, the a_i are indexes, and n is an integer between 1 and 5. This stands for $x[a_1][a_2]\ldots[a_n]$, and returns a `GEN`. The macros `gel` (resp. `gmael`) are synonyms for `gmael1` (resp. `gmael2`).

Finally, the macro `gcoeff`(M, i, j) has exactly the meaning of `M[i,j]` in GP when `M` is a matrix. Note that due to the implementation of `t_MAT`s as horizontal lists of vertical vectors, `gcoeff(x,y)` is actually equivalent to `gmael(y,x)`. One should use `gcoeff` in matrix context, and `gmael` otherwise.

4.2.4 Variations on basic functions. In the library syntax descriptions in Chapter 3, we have only given the basic names of the functions. For example `gadd`(x, y) assumes that x and y are `GEN`s, and *creates* the result $x+y$ on the PARI stack. For most of the basic operators and functions, many other variants are available. We give some examples for `gadd`, but the same is true for all the basic operators, as well as for some simple common functions (a complete list is given in Chapter 6):

`GEN gaddgs(GEN x, long y)`

`GEN gaddsg(long x, GEN y)`

In the following one, `z` is a preexisting `GEN` and the result of the corresponding operation is put into `z`. The size of the PARI stack does not change:

`void gaddz(GEN x, GEN y, GEN z)`

(This last form is inefficient in general and deprecated outside of PARI kernel programming.) Low level kernel functions implement these operators for specialized arguments and are also available: Level 0 deals with operations at the word level (`long`s and `ulong`s), Level 1 with `t_INT` and `t_REAL` and Level 2 with the rest (modular arithmetic, polynomial arithmetic and linear algebra). Here are some examples of Level 1 functions:

`GEN addii(GEN x, GEN y)`: here x and y are `GEN`s of type `t_INT` (this is not checked).

`GEN addrr(GEN x, GEN y)`: here x and y are `GEN`s of type `t_REAL` (this is not checked).

There also exist functions **addir**, **addri**, **mpadd** (whose two arguments can be of type `t_INT` or `t_REAL`), **addis** (to add a `t_INT` and a `long`) and so on.

The Level 1 names are self-explanatory once you know that **i** stands for a `t_INT`, **r** for a `t_REAL`, **mp** for i or r, **s** for a signed C long integer, **u** for an unsigned C long integer; finally the suffix **z** means that the result is not created on the PARI stack but assigned to a preexisting GEN object passed as an extra argument. Chapter 6 gives a description of these low-level functions.

Level 2 names are more complicated, see Section 7.1 for all the gory details, and we content ourselves with a simple example used to implement `t_INTMOD` arithmetic:

`GEN Fp_add(GEN x, GEN y, GEN m)`: returns the sum of x and y modulo m. Here x, y, m are `t_INT`s (this is not checked). The operation is more efficient if the inputs x, y are reduced modulo m, but this is not a necessary condition.

Important Note. These specialized functions are of course more efficient than the generic ones, but note the hidden danger here: the types of the objects involved (which is not checked) must be severely controlled, e.g. using `addii` on a `t_FRAC` argument will cause disasters. Type mismatches may corrupt the PARI stack, though in most cases they will just immediately overflow the stack. Because of this, the PARI philosophy of giving a result which is as exact as possible, enforced for generic functions like `gadd` or `gmul`, is dropped in kernel routines of Level 1, where it is replaced by the much simpler rule: the result is a `t_INT` if and only if all arguments are integer types (`t_INT` but also C `long` and `ulong`) and a `t_REAL` otherwise. For instance, multiplying a `t_REAL` by a `t_INT` always yields a `t_REAL` if you use `mulir`, where `gmul` returns the `t_INT` `gen_0` if the integer is 0.

4.2.5 Portability: 32-bit / 64-bit architectures.

PARI supports both 32-bit and 64-bit based machines, but not simultaneously! The library is compiled assuming a given architecture, and some of the header files you include (through `pari.h`) will have been modified to match the library.

Portable macros are defined to bypass most machine dependencies. If you want your programs to run identically on 32-bit and 64-bit machines, you have to use these, and not the corresponding numeric values, whenever the precise size of your `long` integers might matter. Here are the most important ones:

	64-bit	32-bit	
`BITS_IN_LONG`	64	32	
`LONG_IS_64BIT`	defined	undefined	
`DEFAULTPREC`	3	4	(≈ 19 decimal digits, see formula below)
`MEDDEFAULTPREC`	4	6	(≈ 38 decimal digits)
`BIGDEFAULTPREC`	5	8	(≈ 57 decimal digits)

For instance, suppose you call a transcendental function, such as

`GEN gexp(GEN x, long prec)`.

The last argument `prec` is an integer ≥ 3, corresponding to the default floating point precision required. It is *only* used if `x` is an exact object, otherwise the relative precision is determined by the precision of `x`. Since the parameter `prec` sets the size of the inexact result counted in (`long`) *words* (including codewords), the same value of `prec` will yield different results on 32-bit and 64-bit machines. Real numbers have two codewords (see Section 4.5), so the formula for computing the bit accuracy is

$$\texttt{bit\_accuracy(prec)} = (\texttt{prec} - 2) * \texttt{BITS\_IN\_LONG}$$

(this is actually the definition of an inline function). The corresponding accuracy expressed in decimal digits would be

$$\texttt{bit\_accuracy(prec)} * \log(2)/\log(10).$$

For example if the value of `prec` is 5, the corresponding accuracy for 32-bit machines is $(5 - 2) * \log(2^{32})/\log(10) \approx 28$ decimal digits, while for 64-bit machines it is $(5 - 2) * \log(2^{64})/\log(10) \approx 57$ decimal digits.

Thus, you must take care to change the prec parameter you are supplying according to the bit size, either using the default precisions given by the various DEFAULTPRECs, or by using conditional constructs of the form:

```
#ifndef LONG_IS_64BIT
  prec = 4;
#else
  prec = 6;
#endif
```

which is in this case equivalent to the statement prec = MEDDEFAULTPREC;.

Note that for parity reasons, half the accuracies available on 32-bit architectures (the odd ones) have no precise equivalents on 64-bit machines.

4.2.6 Using malloc / free. You should make use of the PARI stack as much as possible, and avoid allocating objects using the customary functions. If you do, you should use, or at least have a very close look at, the following wrappers:

void* pari_malloc(size_t size) calls malloc to allocate size bytes and returns a pointer to the allocated memory. If the request fails, an error is raised. The SIGINT signal is blocked until malloc returns, to avoid leaving the system stack in an inconsistent state.

void* pari_realloc(void* ptr, size_t size) as pari_malloc but calls realloc instead of malloc.

void pari_realloc_ip(void** ptr, size_t size) equivalent to *ptr= realloc(*ptr, size), while blocking SIGINT.

void* pari_calloc(size_t size) as pari_malloc, setting the memory to zero.

void pari_free(void* ptr) calls free to liberate the memory space pointed to by ptr, which must have been allocated by malloc (pari_malloc) or realloc (pari_realloc). The SIGINT signal is blocked until free returns.

If you use the standard libc functions instead of our wrappers, then your functions will be subtly incompatible with the gp calculator: when the user tries to interrupt a computation, the calculator may crash (if a system call is interrupted at the wrong time).

4.3 Garbage collection.

4.3.1 Why and how.

As we have seen, pari_init allocates a big range of addresses, the *stack*, that are going to be used throughout. Recall that all PARI objects are pointers. Except for a few universal objects, they all point at some part of the stack.

The stack starts at the address bot and ends just before top. This means that the quantity

$$(\texttt{top} - \texttt{bot}) \,/\, \texttt{sizeof(long)}$$

is (roughly) equal to the size argument of pari_init. The PARI stack also has a "current stack pointer" called **avma**, which stands for **av**ailable **m**emory **a**ddress. These three variables are global (declared by pari.h). They are of type pari_sp, which means *pari stack pointer*.

The stack is oriented upside-down: the more recent an object, the closer to `bot`. Accordingly, initially `avma` = `top`, and `avma` gets *decremented* as new objects are created. As its name indicates, `avma` always points just *after* the first free address on the stack, and `(GEN)avma` is always (a pointer to) the latest created object. When `avma` reaches `bot`, the stack overflows, aborting all computations, and an error message is issued. To avoid this *you* need to clean up the stack from time to time, when intermediate objects are not needed anymore. This is called "*garbage collecting.*"

We are now going to describe briefly how this is done. We will see many concrete examples in the next subsection.

• First, PARI routines do their own garbage collecting, which means that whenever a documented function from the library returns, only its result(s) have been added to the stack, possibly up to a very small overhead (undocumented ones may not do this). In particular, a PARI function that does not return a `GEN` does not clutter the stack. Thus, if your computation is small enough (e.g. you call few PARI routines, or most of them return `long` integers), then you do not need to do any garbage collecting. This is probably the case in many of your subroutines. Of course the objects that were on the stack *before* the function call are left alone. Except for the ones listed below, PARI functions only collect their own garbage.

• It may happen that all objects that were created after a certain point can be deleted — for instance, if the final result you need is not a `GEN`, or if some search proved futile. Then, it is enough to record the value of `avma` just *before* the first garbage is created, and restore it upon exit:

```
pari_sp av = avma; /* record initial avma */

garbage ...
set_avma(av); /* restore it */
```

All objects created in the `garbage` zone will eventually be overwritten: they should no longer be accessed after `avma` has been restored. Think of the `set_avma` call as a simple `avma = av` restoring the `avma` value.

• If you want to destroy (i.e. give back the memory occupied by) the *latest* PARI object on the stack (e.g. the latest one obtained from a function call), you can use the function

```
void cgiv(GEN z)
```

where `z` is the object you want to give back. This is equivalent to the above where the initial `av` is computed from `z`.

• Unfortunately life is not so simple, and sometimes you will want to give back accumulated garbage *during* a computation without losing recent data. We shall start with the lowest level function to get a feel for the underlying mechanisms, we shall describe simpler variants later:

`GEN gerepile(pari_sp ltop, pari_sp lbot, GEN q)`. This function cleans up the stack between `ltop` and `lbot`, where `lbot` < `ltop`, and returns the updated object `q`. This means:

1) we translate (copy) all the objects in the interval [`avma`, `lbot`[, so that its right extremity abuts the address `ltop`. Graphically

```
             bot           avma   lbot          ltop     top
End of stack  |-------------[++++++[-/-/-/-/-/-/-|++++++++|  Start
                 free memory            garbage
```

becomes:

```
             bot                         avma   ltop     top
End of stack  |---------------------------[++++++[++++++++|  Start
                        free memory
```

where `++` denote significant objects, `--` the unused part of the stack, and `-/-` the garbage we remove.

2) The function then inspects all the PARI objects between `avma` and `lbot` (i.e. the ones that we want to keep and that have been translated) and looks at every component of such an object which is not a codeword. Each such component is a pointer to an object whose address is either

— between `avma` and `lbot`, in which case it is suitably updated,

— larger than or equal to `ltop`, in which case it does not change, or

— between `lbot` and `ltop` in which case `gerepile` raises an error ("significant pointers lost in gerepile").

3) **avma** is updated (we add `ltop` − `lbot` to the old value).

4) We return the (possibly updated) object `q`: if `q` initially pointed between `avma` and `lbot`, we return the updated address, as in 2). If not, the original address is still valid, and is returned!

As stated above, no component of the remaining objects (in particular `q`) should belong to the erased segment [`lbot`, `ltop`[, and this is checked within `gerepile`. But beware as well that the addresses of the objects in the translated zone change after a call to `gerepile`, so you must not access any pointer which previously pointed into the zone below `ltop`. If you need to recover more than one object, use the `gerepileall` function below.

Remark. As a consequence of the preceding explanation, if a PARI object is to be relocated by gerepile then, apart from universal objects, the chunks of memory used by its components should be in consecutive memory locations. All `GEN`s created by documented PARI functions are guaranteed to satisfy this. This is because the `gerepile` function knows only about *two connected zones*: the garbage that is erased (between `lbot` and `ltop`) and the significant pointers that are copied and updated. If there is garbage interspersed with your objects, disaster occurs when we try to update them and consider the corresponding "pointers". In most cases of course the said garbage is in fact a bunch of other `GEN`s, in which case we simply waste time copying and updating them for nothing. But be wary when you allow objects to become disconnected.

In practice this is achieved by the following programming idiom:

```
ltop = avma; garbage(); lbot = avma; q = anything();
return gerepile(ltop, lbot, q); /* returns the updated q */
```

or directly

```
ltop = avma; garbage(); lbot = avma;
return gerepile(ltop, lbot, anything());
```

Beware that

```
ltop = avma; garbage();
return gerepile(ltop, avma, anything())
```

might work, but should be frowned upon. We cannot predict whether `avma` is evaluated after or before the call to `anything()`: it depends on the compiler. If we are out of luck, it is *after* the call, so the result belongs to the garbage zone and the `gerepile` statement becomes equivalent to `set_avma(ltop)`. Thus we return a pointer to random garbage.

4.3.2 Variants.

`GEN gerepileupto(pari_sp ltop, GEN q)`. Cleans the stack between `ltop` and the *connected* object `q` and returns `q` updated. For this to work, `q` must have been created *before* all its components, otherwise they would belong to the garbage zone! Unless mentioned otherwise, documented PARI functions guarantee this.

`GEN gerepilecopy(pari_sp ltop, GEN x)`. Functionally equivalent to, but more efficient than

```
gerepileupto(ltop, gcopy(x))
```

In this case, the `GEN` parameter `x` need not satisfy any property before the garbage collection: it may be disconnected, components created before the root, and so on. Of course, this is about twice slower than either `gerepileupto` or `gerepile`, because `x` has to be copied to a clean stack zone first. This function is a special case of `gerepileall` below, where $n = 1$.

`void gerepileall(pari_sp ltop, int n, ...)`. To cope with complicated cases where many objects have to be preserved. The routine expects n further arguments, which are the *addresses* of the `GEN`s you want to preserve:

```
pari_sp ltop = avma;
...; y = ...; ... x = ...; ...;
gerepileall(ltop, 2, &x, &y);
```

It cleans up the most recent part of the stack (between `ltop` and `avma`), updating all the `GEN`s added to the argument list. A copy is done just before the cleaning to preserve them, so they do not need to be connected before the call. With `gerepilecopy`, this is the most robust of the `gerepile` functions (the less prone to user error), hence the slowest.

`void gerepileallsp(pari_sp ltop, pari_sp lbot, int n, ...)`. More efficient, but trickier than `gerepileall`. Cleans the stack between `lbot` and `ltop` and updates the `GEN`s pointed at by the elements of `gptr` without any further copying. This is subject to the same restrictions as `gerepile`, the only difference being that more than one address gets updated.

4.3.3 Examples.

4.3.3.1 gerepile.

Let `x` and `y` be two preexisting PARI objects and suppose that we want to compute $x^2 + y^2$. This is done using the following program:

```
GEN x2 = gsqr(x);
GEN y2 = gsqr(y), z = gadd(x2,y2);
```

The `GEN z` indeed points at the desired quantity. However, consider the stack: it contains as unnecessary garbage `x2` and `y2`. More precisely it contains (in this order) `z`, `y2`, `x2`. (Recall that, since the stack grows downward from the top, the most recent object comes first.)

It is not possible to get rid of x2, y2 before z is computed, since they are used in the final operation. We cannot record avma before x2 is computed and restore it later, since this would destroy z as well. It is not possible either to use the function cgiv since x2 and y2 are not at the bottom of the stack and we do not want to give back z.

But using gerepile, we can give back the memory locations corresponding to x2, y2, and move the object z upwards so that no space is lost. Specifically:

```
pari_sp ltop = avma;  /* remember the current top of the stack */
GEN x2 = gsqr(x);
GEN y2 = gsqr(y);
pari_sp lbot = avma;  /* the bottom of the garbage pile */
GEN z = gadd(x2, y2); /* z is now the last object on the stack */
z = gerepile(ltop, lbot, z);
```

Of course, the last two instructions could also have been written more simply:

```
z = gerepile(ltop, lbot, gadd(x2,y2));
```

In fact gerepileupto is even simpler to use, because the result of gadd is the last object on the stack and gadd is guaranteed to return an object suitable for gerepileupto:

```
ltop = avma;
z = gerepileupto(ltop, gadd(gsqr(x), gsqr(y)));
```

Make sure you understand exactly what has happened before you go on!

Remark on assignments and gerepile. When the tree structure and the size of the PARI objects which will appear in a computation are under control, one may allocate sufficiently large objects at the beginning, use assignment statements, then simply restore avma. Coming back to the above example, note that *if* we know that x and y are of type real fitting into DEFAULTPREC words, we can program without using gerepile at all:

```
z = cgetr(DEFAULTPREC); ltop = avma;
gaffect(gadd(gsqr(x), gsqr(y)), z);
set_avma(ltop);
```

This is often *slower* than a craftily used gerepile though, and certainly more cumbersome to use. As a rule, assignment statements should generally be avoided.

Variations on a theme. it is often necessary to do several gerepiles during a computation. However, the fewer the better. The only condition for gerepile to work is that the garbage be connected. If the computation can be arranged so that there is a minimal number of connected pieces of garbage, then it should be done that way.

For example suppose we want to write a function of two GEN variables x and y which creates the vector $[x^2 + y, y^2 + x]$. Without garbage collecting, one would write:

```
p1 = gsqr(x); p2 = gadd(p1, y);
p3 = gsqr(y); p4 = gadd(p3, x);
z = mkvec2(p2, p4);  /* not suitable for gerepileupto! */
```

This leaves a dirty stack containing (in this order) z, p4, p3, p2, p1. The garbage here consists of p1 and p3, which are separated by p2. But if we compute p3 *before* p2 then the garbage becomes connected, and we get the following program with garbage collecting:

```
ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(p1,y);
gel(z, 2) = gadd(p3,x); z = gerepile(ltop,lbot,z);
```

Finishing by `z = gerepileupto(ltop, z)` would be ok as well. Beware that

```
ltop = avma; p1 = gadd(gsqr(x), y); p3 = gadd(gsqr(y), x);
z = cgetg(3, t_VEC);
gel(z, 1) = p1;
gel(z, 2) = p3; z = gerepileupto(ltop,z); /* WRONG */
```

is a disaster since `p1` and `p3` are created before `z`, so the call to `gerepileupto` overwrites them, leaving `gel(z, 1)` and `gel(z, 2)` pointing at random data! The following does work:

```
ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = mkvec2(gadd(p1,y), gadd(p3,x));
z = gerepile(ltop,lbot,z);
```

but is very subtly wrong in the sense that `z = gerepileupto(ltop, z)` would *not* work. The reason being that `mkvec2` creates the root `z` of the vector *after* its arguments have been evaluated, creating the components of `z` too early; `gerepile` does not care, but the created `z` is a time bomb which will explode on any later `gerepileupto`. On the other hand

```
ltop = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(gsqr(x), y);
gel(z, 2) = gadd(gsqr(y), x); z = gerepileupto(ltop,z); /* INEFFICIENT */
```

leaves the results of `gsqr(x)` and `gsqr(y)` on the stack (and lets `gerepileupto` update them for naught). Finally, the most elegant and efficient version (with respect to time and memory use) is as follows

```
z = cgetg(3, t_VEC);
ltop = avma; gel(z, 1) = gerepileupto(ltop, gadd(gsqr(x), y));
ltop = avma; gel(z, 2) = gerepileupto(ltop, gadd(gsqr(y), x));
```

which avoids updating the container `z` and cleans up its components individually, as soon as they are computed.

One last example. Let us compute the product of two complex numbers x and y, using the $3M$ method which requires 3 multiplications instead of the obvious 4. Let $z = x*y$, and set $x = x_r + i*x_i$ and similarly for y and z. We compute $p_1 = x_r * y_r$, $p_2 = x_i * y_i$, $p_3 = (x_r + x_i) * (y_r + y_i)$, and then we have $z_r = p_1 - p_2$, $z_i = p_3 - (p_1 + p_2)$. The program is as follows:

```
ltop = avma;
p1 = gmul(gel(x,1), gel(y,1));
p2 = gmul(gel(x,2), gel(y,2));
p3 = gmul(gadd(gel(x,1), gel(x,2)), gadd(gel(y,1), gel(y,2)));
p4 = gadd(p1,p2);
lbot = avma; z = cgetg(3, t_COMPLEX);
gel(z, 1) = gsub(p1,p2);
gel(z, 2) = gsub(p3,p4); z = gerepile(ltop,lbot,z);
```

Exercise. Write a function which multiplies a matrix by a column vector. Hint: start with a `cgetg` of the result, and use `gerepile` whenever a coefficient of the result vector is computed. You can look at the answer in `src/basemath/RgV.c:RgM_RgC_mul()`.

4.3.3.2 gerepileall.

Let us now see why we may need the `gerepileall` variants. Although it is not an infrequent occurrence, we do not give a specific example but a general one: suppose that we want to do a computation (usually inside a larger function) producing more than one PARI object as a result, say two for instance. Then even if we set up the work properly, before cleaning up we have a stack which has the desired results `z1`, `z2` (say), and then connected garbage from lbot to ltop. If we write

```
z1 = gerepile(ltop, lbot, z1);
```

then the stack is cleaned, the pointers fixed up, but we have lost the address of `z2`. This is where we need the gerepileall function:

```
gerepileall(ltop, 2, &z1, &z2)
```

copies `z1` and `z2` to new locations, cleans the stack from `ltop` to the old `avma`, and updates the pointers `z1` and `z2`. Here we do not assume anything about the stack: the garbage can be disconnected and `z1`, `z2` need not be at the bottom of the stack. If all of these assumptions are in fact satisfied, then we can call `gerepilemanysp` instead, which is usually faster since we do not need the initial copy (on the other hand, it is less cache friendly).

A most important usage is "random" garbage collection during loops whose size requirements we cannot (or do not bother to) control in advance:

```
pari_sp av = avma;
GEN x, y;
while (...)
{
  garbage(); x = anything();
  garbage(); y = anything(); garbage();
  if (gc_needed(av,1)) /* memory is running low (half spent since entry) */
    gerepileall(av, 2, &x, &y);
}
```

Here we assume that only `x` and `y` are needed from one iteration to the next. As it would be costly to call gerepile once for each iteration, we only do it when it seems to have become necessary.

More precisely, the macro `stack_lim(av,`n`)` denotes an address where $2^{n-1}/(2^{n-1}+1)$ of the remaining stack space since reference point `av` is exhausted ($1/2$ for $n = 1$, $2/3$ for $n = 2$). The test `gc_needed(av,`n`)` becomes true whenever `avma` drops below that address.

4.3.4 Comments.

First, `gerepile` has turned out to be a flexible and fast garbage collector for number-theoretic computations, which compares favorably with more sophisticated methods used in other systems. Our benchmarks indicate that the price paid for using `gerepile` and `gerepile`-related copies, when properly used, is usually less than 1% of the total running time, which is quite acceptable!

Second, it is of course harder on the programmer, and quite error-prone if you do not stick to a consistent PARI programming style. If all seems lost, just use `gerepilecopy` (or `gerepileall`) to fix up the stack for you. You can always optimize later when you have sorted out exactly which routines are crucial and what objects need to be preserved and their usual sizes.

If you followed us this far, congratulations, and rejoice: the rest is much easier.

4.4 Creation of PARI objects, assignments, conversions.

4.4.1 Creation of PARI objects. The basic function which creates a PARI object is

`GEN cgetg(long l, long t)` l specifies the number of longwords to be allocated to the object, and t is the type of the object, in symbolic form (see Section 4.5 for the list of these). The precise effect of this function is as follows: it first creates on the PARI *stack* a chunk of memory of size `length` longwords, and saves the address of the chunk which it will in the end return. If the stack has been used up, a message to the effect that "the PARI stack overflows" is printed, and an error raised. Otherwise, it sets the type and length of the PARI object. In effect, it fills its first codeword (`z[0]`). Many PARI objects also have a second codeword (types `t_INT`, `t_REAL`, `t_PADIC`, `t_POL`, and `t_SER`). In case you want to produce one of those from scratch, which should be exceedingly rare, *it is your responsibility to fill this second codeword*, either explicitly (using the macros described in Section 4.5), or implicitly using an assignment statement (using `gaffect`).

Note that the length argument l is predetermined for a number of types: 3 for types `t_INTMOD`, `t_FRAC`, `t_COMPLEX`, `t_POLMOD`, `t_RFRAC`, 4 for type `t_QUAD`, and 5 for type `t_PADIC` and `t_QFB`. However for the sake of efficiency, `cgetg` does not check this: disasters will occur if you give an incorrect length for those types.

Notes. 1) The main use of this function is create efficiently a constant object, or to prepare for later assignments (see Section 4.4.3). Most of the time you will use `GEN` objects as they are created and returned by PARI functions. In this case you do not need to use `cgetg` to create space to hold them.

2) For the creation of leaves, i.e. `t_INT` or `t_REAL`,

`GEN cgeti(long length)`

`GEN cgetr(long length)`

should be used instead of `cgetg(length, t_INT)` and `cgetg(length, t_REAL)` respectively. Finally

`GEN cgetc(long prec)`

creates a `t_COMPLEX` whose real and imaginary part are `t_REAL`s allocated by `cgetr(prec)`.

Examples. 1) Both `z = cgeti(DEFAULTPREC)` and `cgetg(DEFAULTPREC, t_INT)` create a `t_INT` whose "precision" is `bit_accuracy(DEFAULTPREC)` = 64. This means `z` can hold rational integers of absolute value less than 2^{64}. Note that in both cases, the second codeword is *not* filled. Of course we could use numerical values, e.g. `cgeti(4)`, but this would have different meanings on different machines as `bit_accuracy(4)` equals 64 on 32-bit machines, but 128 on 64-bit machines.

2) The following creates a *complex number* whose real and imaginary parts can hold real numbers of precision `bit_accuracy(MEDDEFAULTPREC)` = 96 bits:

```
z = cgetg(3, t_COMPLEX);
gel(z, 1) = cgetr(MEDDEFAULTPREC);
gel(z, 2) = cgetr(MEDDEFAULTPREC);
```

or simply `z = cgetc(MEDDEFAULTPREC)`.

3) To create a matrix object for 4×3 matrices:

```
z = cgetg(4, t_MAT);
for(i=1; i<4; i++) gel(z, i) = cgetg(5, t_COL);
```

or simply `z = zeromatcopy(4, 3)`, which further initializes all entries to `gen_0`.

These last two examples illustrate the fact that since PARI types are recursive, all the branches of the tree must be created. The function **cgetg** creates only the "root", and other calls to `cgetg` must be made to produce the whole tree. For matrices, a common mistake is to think that `z = cgetg(4, t_MAT)` (for example) creates the root of the matrix: one needs also to create the column vectors of the matrix (obviously, since we specified only one dimension in the first `cgetg`!). This is because a matrix is really just a row vector of column vectors (hence a priori not a basic type), but it has been given a special type number so that operations with matrices become possible.

Finally, to facilitate input of constant objects when speed is not paramount, there are four `varargs` functions:

`GEN mkintn(long n, ...)` returns the nonnegative `t_INT` whose development in base 2^{32} is given by the following n 32bit-words (`unsigned int`).

```
mkintn(3, a2, a1, a0);
```

returns $a_2 2^{64} + a_1 2^{32} + a_0$.

`GEN mkpoln(long n, ...)` Returns the `t_POL` whose n coefficients (`GEN`) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial $X^2 + 2X$ (in variable 0, use `setvarn` if you want other variable numbers). Beware that n is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose n coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose n coefficients (`GEN`) follow.

Warning. Contrary to the policy of general PARI functions, the latter three functions do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. For instance

```
/* gerepile-safe: components are universal objects */
z = mkvecn(3, gen_1, gen_0, gen_2);
/* not OK for gerepileupto: stoi(3) creates component before root */
z = mkvecn(3, stoi(3), gen_0, gen_2);
/* NO! First vector component x is destroyed */
x = gclone(gen_1);
z = mkvecn(3, x, gen_0, gen_2);
gunclone(x);
```

The following function is also available as a special case of `mkintn`:

`GEN uu32toi(ulong a, ulong b)`

Returns the `GEN` equal to $2^{32}a + b$, *assuming* that $a, b < 2^{32}$. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

4.4.2 Sizes.

`long gsizeword(GEN x)` returns the total number of `BITS_IN_LONG`-bit words occupied by the tree representing `x`.

`long gsizebyte(GEN x)` returns the total number of bytes occupied by the tree representing `x`, i.e. `gsizeword(x)` multiplied by `sizeof(long)`. This is normally useless since PARI functions use a number of *words* as input for lengths and precisions.

4.4.3 Assignments. Firstly, if `x` and `y` are both declared as `GEN` (i.e. pointers to something), the ordinary C assignment `y = x` makes perfect sense: we are just moving a pointer around. However, physically modifying either `x` or `y` (for instance, `x[1] = 0`) also changes the other one, which is usually not desirable.

Very important note. Using the functions described in this paragraph is inefficient and often awkward: one of the `gerepile` functions (see Section 4.3) should be preferred. See the paragraph end for one exception to this rule.

The general PARI assignment function is the function **gaffect** with the following syntax:

`void gaffect(GEN x, GEN y)`

Its effect is to assign the PARI object `x` into the *preexisting* object `y`. Both `x` and `y` must be *scalar* types. For convenience, vector or matrices of scalar types are also allowed.

This copies the whole structure of `x` into `y` so many conditions must be met for the assignment to be possible. For instance it is allowed to assign a `t_INT` into a `t_REAL`, but the converse is forbidden. For that, you must use the truncation or rounding function of your choice, e.g.`mpfloor`.

It can also happen that `y` is not large enough or does not have the proper tree structure to receive the object `x`. For instance, let `y` the zero integer with length equal to 2; then `y` is too small to accommodate any nonzero `t_INT`. In general common sense tells you what is possible, keeping in mind the PARI philosophy which says that if it makes sense it is valid. For instance, the assignment of an imprecise object into a precise one does *not* make sense. However, a change in precision of imprecise objects is allowed, even if it *increases* its accuracy: we complement the "mantissa" with infinitely many 0 digits in this case. (Mantissa between quotes, because this is not restricted to `t_REAL`s, it also applies for p-adics for instance.)

All functions ending in “`z`” such as **gaddz** (see Section 4.2.4) implicitly use this function. In fact what they exactly do is record **avma** (see Section 4.3), perform the required operation, **gaffect** the result to the last operand, then restore the initial `avma`.

You can assign ordinary C long integers into a PARI object (not necessarily of type `t_INT`) using

`void gaffsg(long s, GEN y)`

Note. Due to the requirements mentioned above, it is usually a bad idea to use `gaffect` statements. There is one exception: for simple objects (e.g. leaves) whose size is controlled, they can be easier to use than `gerepile`, and about as efficient.

Coercion. It is often useful to coerce an inexact object to a given precision. For instance at the beginning of a routine where precision can be kept to a minimum; otherwise the precision of the input is used in all subsequent computations, which is inefficient if the latter is known to thousands of digits. One may use the `gaffect` function for this, but it is easier and more efficient to call

`GEN gtofp(GEN x, long prec)` converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of length `prec`.

4.4.4 Copy. It is also very useful to copy a PARI object, not just by moving around a pointer as in the `y = x` example, but by creating a copy of the whole tree structure, without pre-allocating a possibly complicated `y` to use with `gaffect`. The function which does this is called **gcopy**. Its syntax is:

`GEN gcopy(GEN x)`

and the effect is to create a new copy of x on the PARI stack.

Sometimes, on the contrary, a quick copy of the skeleton of `x` is enough, leaving pointers to the original data in `x` for the sake of speed instead of making a full recursive copy. Use `GEN shallowcopy(GEN x)` for this. Note that the result is not suitable for `gerepileupto` !

Make sure at this point that you understand the difference between `y = x`, `y = gcopy(x)`, `y = shallowcopy(x)` and `gaffect(x,y)`.

4.4.5 Clones. Sometimes, it is more efficient to create a *persistent* copy of a PARI object. This is not created on the stack but on the heap, hence unaffected by `gerepile` and friends. The function which does this is called **gclone**. Its syntax is:

`GEN gclone(GEN x)`

A clone can be removed from the heap (thus destroyed) using

`void gunclone(GEN x)`

No PARI object should keep references to a clone which has been destroyed!

4.4.6 Conversions. The following functions convert C objects to PARI objects (creating them on the stack as usual):

`GEN stoi(long s)`: C `long` integer ("small") to `t_INT`.

`GEN dbltor(double s)`: C `double` to `t_REAL`. The accuracy of the result is 19 decimal digits, i.e. a type `t_REAL` of length `DEFAULTPREC`, although on 32-bit machines only 16 of them are significant.

We also have the converse functions:

`long itos(GEN x)`: `x` must be of type `t_INT`,

`double rtodbl(GEN x)`: `x` must be of type `t_REAL`,

as well as the more general ones:

`long gtolong(GEN x)`,

`double gtodouble(GEN x)`.

4.5 Implementation of the PARI types.

We now go through each type and explain its implementation. Let `z` be a `GEN`, pointing at a PARI object. In the following paragraphs, we will constantly mix two points of view: on the one hand, `z` is treated as the C pointer it is, on the other, as PARI's handle on some mathematical entity, so we will shamelessly write $z \neq 0$ to indicate that the *value* thus represented is nonzero (in which case the *pointer* `z` is certainly not `NULL`). We offer no apologies for this style. In fact, you had better feel comfortable juggling both views simultaneously in your mind if you want to write correct PARI programs.

Common to all the types is the first codeword `z[0]`, which we do not have to worry about since this is taken care of by `cgetg`. Its precise structure depends on the machine you are using, but it always contains the following data: the *internal type number* attached to the symbolic type name, the *length* of the root in longwords, and a technical bit which indicates whether the object is a clone or not (see Section 4.4.5). This last one is used by `gp` for internal garbage collecting, you will not have to worry about it.

Some types have a second codeword, different for each type, which we will soon describe as we will shortly consider each of them in turn.

The first codeword is handled through the following *macros*:

`long typ(GEN z)` returns the type number of `z`.

`void settyp(GEN z, long n)` sets the type number of `z` to `n` (you should not have to use this function if you use `cgetg`).

`long lg(GEN z)` returns the length (in longwords) of the root of `z`.

`long setlg(GEN z, long l)` sets the length of `z` to `l`; you should not have to use this function if you use `cgetg`.

`void lg_increase(GEN z)` increase the length of `z` by 1; you should not have to use this function if you use `cgetg`.

`long isclone(GEN z)` is `z` a clone?

`void setisclone(GEN z)` sets the *clone* bit.

`void unsetisclone(GEN z)` clears the *clone* bit.

Important remark. For the sake of efficiency, none of the codeword-handling macros check the types of their arguments even when there are stringent restrictions on their use. It is trivial to create invalid objects, or corrupt one of the "universal constants" (e.g. setting the sign of `gen_0` to 1), and they usually provide negligible savings. Use higher level functions whenever possible.

Remark. The clone bit is there so that `gunclone` can check it is deleting an object which was allocated by `gclone`. Miscellaneous vector entries are often cloned by `gp` so that a GP statement like `v[1] = x` does not involve copying the whole of `v`: the component `v[1]` is deleted if its clone bit is set, and is replaced by a clone of `x`. Don't set/unset yourself the clone bit unless you know what you are doing: in particular *never* set the clone bit of a vector component when the said vector is scheduled to be uncloned. Hackish code may abuse the clone bit to tag objects for reasons unrelated to the above instead of using proper data structures. Don't do that.

4.5.1 Type `t_INT` (integer). this type has a second codeword `z[1]` which contains the following information:

the sign of `z`: coded as 1, 0 or -1 if $\texttt{z} > 0$, $\texttt{z} = 0$, $\texttt{z} < 0$ respectively.

the *effective length* of `z`, i.e. the total number of significant longwords. This means the following: apart from the integer 0, every integer is "normalized", meaning that the most significant mantissa longword is nonzero. However, the integer may have been created with a longer length. Hence the "length" which is in `z[0]` can be larger than the "effective length" which is in `z[1]`.

This information is handled using the following macros:

`long signe(GEN z)` returns the sign of `z`.

`void setsigne(GEN z, long s)` sets the sign of `z` to `s`.

`long lgefint(GEN z)` returns the effective length of `z`.

`void setlgefint(GEN z, long l)` sets the effective length of `z` to `l`.

The integer 0 can be recognized either by its sign being 0, or by its effective length being equal to 2. Now assume that $\texttt{z} \neq 0$, and let

$$|z| = \sum_{i=0}^{n} z_i B^i, \quad \text{where } z_n \neq 0 \text{ and } B = 2^{\texttt{BITS\_IN\_LONG}}.$$

With these notations, n is `lgefint(z) - 3`, and the mantissa of `z` may be manipulated via the following interface:

`GEN int_MSW(GEN z)` returns a pointer to the most significant word of `z`, z_n.

`GEN int_LSW(GEN z)` returns a pointer to the least significant word of `z`, z_0.

`GEN int_W(GEN z, long i)` returns the i-th significant word of `z`, z_i. Accessing the i-th significant word for $i > n$ yields unpredictable results.

`GEN int_W_lg(GEN z, long i, long lz)` returns the i-th significant word of `z`, z_i, assuming `lgefint(z)` is `lz` ($= n + 3$). Accessing the i-th significant word for $i > n$ yields unpredictable results.

`GEN int_precW(GEN z)` returns the previous (less significant) word of `z`, z_{i-1} assuming `z` points to z_i.

`GEN int_nextW(GEN z)` returns the next (more significant) word of `z`, z_{i+1} assuming `z` points to z_i.

Unnormalized integers, such that z_n is possibly 0, are explicitly forbidden. To enforce this, one may write an arbitrary mantissa then call

`void int_normalize(GEN z, long known0)`

normalizes in place a nonnegative integer (such that z_n is possibly 0), assuming at least the first `known0` words are zero.

For instance a binary `and` could be implemented in the following way:

```
GEN AND(GEN x, GEN y) {
  long i, lx, ly, lout;
  long *xp, *yp, *outp; /* mantissa pointers */
  GEN out;

  if (!signe(x) || !signe(y)) return gen_0;
  lx = lgefint(x); xp = int_LSW(x);
  ly = lgefint(y); yp = int_LSW(y); lout = min(lx,ly); /* > 2 */

  out = cgeti(lout); out[1] = evalsigne(1) | evallgefint(lout);
  outp = int_LSW(out);
  for (i=2; i < lout; i++)
  {
    *outp = (*xp) & (*yp);
    outp  = int_nextW(outp);
    xp    = int_nextW(xp);
    yp    = int_nextW(yp);
  }
  if ( !*int_MSW(out) ) out = int_normalize(out, 1);
  return out;
}
```

This low-level interface is mandatory in order to write portable code since PARI can be compiled using various multiprecision kernels, for instance the native one or GNU MP, with incompatible internal structures (for one thing, the mantissa is oriented in different directions).

4.5.2 Type `t_REAL` (real number). this type has a second codeword z[1] which also encodes its sign, obtained or set using the same functions as for a `t_INT`, and a binary exponent. This exponent is handled using the following macros:

`long expo(GEN z)` returns the exponent of `z`. This is defined even when `z` is equal to zero.

`void setexpo(GEN z, long e)` sets the exponent of `z` to `e`.

Note the functions:

`long gexpo(GEN z)` which tries to return an exponent for `z`, even if `z` is not a real number.

`long gsigne(GEN z)` which returns a sign for `z`, even when `z` is a real number of type `t_INT`, `t_FRAC` or `t_REAL`, an infinity (`t_INFINITY`) or a `t_QUAD` of positive discriminant.

The real zero is characterized by having its sign equal to 0. If `z` is not equal to 0, then it is represented as $2^e M$, where e is the exponent, and $M \in [1, 2[$ is the mantissa of z, whose digits are stored in $\texttt{z}[2], \dots, \texttt{z}[\texttt{lg}(\texttt{z}) - 1]$. For historical reasons, the `prec` parameter attached to floating point functions is measured in `BITS_IN_LONG`-bit words and is equal to the length of `x`: yes, this includes the two code words and depends on `sizeof(long)`. For clarity we advise to use `bit_accuracy`, which computes the true length of the mantissa in bits, and convert between bits and `prec` using the `prec2nbits` and `nbits2prec` macros. But keep in mind that the accuracy of `t_REAL` actually increases by increments of `BITS_IN_LONG`bits.

More precisely, let m be the integer (`z[2]`,..., `z[lg(z)-1]`) in base `2^BITS_IN_LONG`; here, `z[2]` is the most significant longword and is normalized, i.e. its most significant bit is 1. Then we have $M := m/2^{\texttt{bit\_accuracy(lg(z))}-1-\texttt{expo}(z)}$.

`GEN mantissa_real(GEN z, long *e)` returns the mantissa m of z, and sets `*e` to the exponent $\texttt{bit\_accuracy}(\texttt{lg(z)}) - 1 - \texttt{expo}(z)$, so that $z = m/2^e$.

Thus, the real number 3.5 to accuracy `bit_accuracy(lg(z))` is represented as `z[0]` (encoding `type` = `t_REAL`, `lg(z)`), `z[1]` (encoding `sign` = 1, `expo` = 1), `z[2]` = `0xe0000000`, `z[3]` = ... = `z[lg(z) - 1]` = `0x0`.

4.5.3 Type `t_INTMOD`. `z[1]` points to the modulus, and `z[2]` at the number representing the class `z`. Both are separate `GEN` objects, and both must be `t_INT`s, satisfying the inequality $0 \le \texttt{z}[2] < \texttt{z}[1]$.

4.5.4 Type `t_FRAC` (rational number). `z[1]` points to the numerator n, and `z[2]` to the denominator d. Both must be of type `t_INT` such that $n \neq 0$, $d > 0$ and $(n, d) = 1$.

4.5.5 Type `t_FFELT` (finite field element). (Experimental)

Components of this type should normally not be accessed directly. Instead, finite field elements should be created using `ffgen`.

The second codeword `z[1]` determines the storage format of the element, among

- `t_FF_FpXQ`: `A=z[2]` and `T=z[3]` are `FpX`, `p=z[4]` is a `t_INT`, where p is a prime number, T is irreducible modulo p, and $\deg A < \deg T$. This represents the element $A \pmod T$ in $\mathbf{F}_p[X]/T$.

- `t_FF_Flxq`: `A=z[2]` and `T=z[3]` are `Flx`, `l=z[4]` is a `t_INT`, where l is a prime number, T is irreducible modulo l, and $\deg A < \deg T$ This represents the element $A \pmod T$ in $\mathbf{F}_l[X]/T$.

- `t_FF_F2xq`: `A=z[2]` and `T=z[3]` are `F2x`, `l=z[4]` is the `t_INT` 2, T is irreducible modulo 2, and $\deg A < \deg T$. This represents the element $A \pmod T$ in $\mathbf{F}_2[X]/T$.

4.5.6 Type `t_COMPLEX` (complex number). `z[1]` points to the real part, and `z[2]` to the imaginary part. The components `z[1]` and `z[2]` must be of type `t_INT`, `t_REAL` or `t_FRAC`. For historical reasons `t_INTMOD` and `t_PADIC` are also allowed (the latter for $p = 2$ or congruent to 3 mod 4 only), but one should rather use the more general `t_POLMOD` construction.

4.5.7 Type `t_PADIC` (p-adic numbers). this type has a second codeword `z[1]` which contains the following information: the p-adic precision (the exponent of p modulo which the p-adic unit corresponding to `z` is defined if `z` is not 0), i.e. one less than the number of significant p-adic digits, and the exponent of `z`. This information can be handled using the following functions:

`long precp(GEN z)` returns the p-adic precision of `z`. This is 0 if $\texttt{z} = 0$.

`void setprecp(GEN z, long l)` sets the p-adic precision of `z` to `l`.

`long valp(GEN z)` returns the p-adic valuation of `z` (i.e. the exponent). This is defined even if `z` is equal to 0.

`void setvalp(GEN z, long e)` sets the p-adic valuation of `z` to `e`.

In addition to this codeword, `z[2]` points to the prime p, `z[3]` points to $p^{\text{precp(z)}}$, and `z[4]` points to a`t_INT` representing the p-adic unit attached to `z` modulo `z[3]` (and to zero if `z` is zero). To summarize, if $z \neq 0$, we have the equality:

$$\mathtt{z} = p^{\text{valp(z)}} * (\mathtt{z}[4] + O(\mathtt{z}[3])), \quad \text{where} \quad \mathtt{z}[3] = O(p^{\text{precp(z)}}).$$

4.5.8 Type `t_QUAD` (quadratic number). `z[1]` points to the canonical polynomial P defining the quadratic field (as output by `quadpoly`), `z[2]` to the "real part" and `z[3]` to the "imaginary part". The latter are of type `t_INT`, `t_FRAC`, `t_INTMOD`, or `t_PADIC` and are to be taken as the coefficients of `z` with respect to the canonical basis $(1, X)$ of $\mathbf{Q}[X]/(P(X))$. Exact complex numbers may be implemented as quadratics, but `t_COMPLEX` is in general more versatile (`t_REAL` components are allowed) and more efficient.

Operations involving a `t_QUAD` and `t_COMPLEX` are implemented by converting the `t_QUAD` to a `t_REAL` (or `t_COMPLEX` with `t_REAL` components) to the accuracy of the `t_COMPLEX`. As a consequence, operations between `t_QUAD` and *exact* `t_COMPLEX`s are not allowed.

4.5.9 Type `t_POLMOD` (polmod). as for `t_INTMOD`s, `z[1]` points to the modulus, and `z[2]` to a polynomial representing the class of `z`. Both must be of type `t_POL` in the same variable, satisfying the inequality $\deg \mathtt{z}[2] < \deg \mathtt{z}[1]$. However, `z[2]` is allowed to be a simplification of such a polynomial, e.g. a scalar. This is tricky considering the hierarchical structure of the variables; in particular, a polynomial in variable of *lesser* priority (see Section 4.6) than the modulus variable is valid, since it is considered as the constant term of a polynomial of degree 0 in the correct variable. On the other hand a variable of *greater* priority is not acceptable.

4.5.10 Type `t_POL` (polynomial). this type has a second codeword. It contains a "*sign*": 0 if the polynomial is equal to 0, and 1 if not (see however the important remark below) and a *variable number* (e.g. 0 for x, 1 for y, etc...).

These data can be handled with the following macros: **signe** and **setsigne** as for `t_INT` and `t_REAL`,

`long varn(GEN z)` returns the variable number of the object `z`,

`void setvarn(GEN z, long v)` sets the variable number of `z` to `v`.

The variable numbers encode the relative priorities of variables, we will give more details in Section 4.6. Note also the function `long gvar(GEN z)` which tries to return a variable number for `z`, even if `z` is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE`, which has lower priority than any other variable number.

The components `z[2]`, `z[3]`,...`z[lg(z)-1]` point to the coefficients of the polynomial *in ascending order*, with `z[2]` being the constant term and so on.

For a `t_POL` of nonzero sign, `degpol`, `leading_coeff`, `constant_coeff`, return its degree, and a pointer to the leading, resp. constant, coefficient with respect to the main variable. Note that no copy is made on the PARI stack so the returned value is not safe for a basic `gerepile` call. Applied to any other type than `t_POL`, the result is unspecified. Those three functions are still defined when the sign is 0, see Section 5.2.7 and Section 10.6.

`long degree(GEN x)` returns the degree of `x` with respect to its main variable even when `x` is not a polynomial (a rational function for instance). By convention, the degree of a zero polynomial is -1.

Important remark. The leading coefficient of a `t_POL` may be equal to zero:

• it is not allowed to be an exact rational 0, such as `gen_0`;

• an exact nonrational 0, like `Mod(0,2)`, is possible for constant polynomials, i.e. of length 3 and no other coefficient: this carries information about the base ring for the polynomial;

• an inexact 0, like `0.E-38` or `O(3^5)`, is always possible. Inexact zeroes do not correspond to an actual 0, but to a very small coefficient according to some metric; we keep them to give information on how much cancellation occurred in previous computations.

A polynomial disobeying any of these rules is an invalid *unnormalized* object. We advise *not* to use low-level constructions to build a `t_POL` coefficient by coefficient, such as

```
GEN T = cgetg(4, t_POL);
T[1] = evalvarn(0);
gel(T, 2) = x;
gel(T, 3) = y;
```

But if you do and it is not clear whether the result will be normalized, call

`GEN normalizepol(GEN x)` applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. This functions sets `signe` (to 0 or 1) properly.

Caveat. A consequence of the remark above is that zero polynomials are characterized by the fact that their sign is 0. It is in general incorrect to check whether `lg(x)` is 2 or `degpol(x)` < 0, although both tests are valid when the coefficient types are under control: for instance, when they are guaranteed to be `t_INT`s or `t_FRAC`s. The same remark applies to `t_SER`s.

4.5.11 Type `t_SER` (power series). This type also has a second codeword, which encodes a "*sign*", i.e. 0 if the power series is 0, and 1 if not, a *variable number* as for polynomials, and an *exponent*. This information can be handled with the following functions: **signe**, **setsigne**, **varn**, **setvarn** as for polynomials, and **valp**, **setvalp** for the exponent as for p-adic numbers. Beware: do *not* use **expo** and **setexpo** on power series.

The coefficients `z[2]`, `z[3]`,...`z[lg(z)-1]` point to the coefficients of `z` in ascending order. As for polynomials (see remark there), the sign of a `t_SER` is 0 if and only all its coefficients are equal to 0. (The leading coefficient cannot be an integer 0.) A series whose coefficients are integers equal to zero is represented as $O(x^n)$ (`zeroser`(vx, n)). A series whose coefficients are exact zeroes, but not all of them integers (e.g. an `t_INTMOD` such as `Mod(0,2)`) is represented as $z * x^{n-1} + O(x^n)$, where z is the 0 of the base ring, as per `Rg_get_0`.

Note that the exponent of a power series can be negative, i.e. we are then dealing with a Laurent series (with a finite number of negative terms).

4.5.12 Type `t_RFRAC` (rational function). `z[1]` points to the numerator n, and `z[2]` on the denominator d. The denominator must be of type `t_POL`, with variable of higher priority than the numerator. The numerator n is not an exact 0 and $(n, d) = 1$ (see `gred_rfac2`).

4.5.13 Type `t_QFB` (binary quadratic form). `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form, and `z[4]` point to the form discriminant. All four are of type `t_INT`.

4.5.14 Type `t_VEC` and `t_COL` (vector). `z[1]`, `z[2]`,... `z[lg(z)-1]` point to the components of the vector.

4.5.15 Type `t_MAT` (matrix). `z[1]`, `z[2]`,... `z[lg(z)-1]` point to the column vectors of `z`, i.e. they must be of type `t_COL` and of the same length.

4.5.16 Type `t_VECSMALL` (vector of small integers). `z[1]`, `z[2]`,... `z[lg(z)-1]` are ordinary signed long integers. This type is used instead of a `t_VEC` of `t_INT`s for efficiency reasons, for instance to implement efficiently permutations, polynomial arithmetic and linear algebra over small finite fields, etc.

4.5.17 Type `t_STR` (character string).

`char * GSTR(z)` (= `(z+1)`) points to the first character of the (`NULL`-terminated) string.

4.5.18 Type `t_ERROR` (error context). This type holds error messages, as well as details about the error, as returned by the exception handling system. The second codeword `z[1]` contains the error type (an `int`, as passed to `pari_err`). The subsequent words `z[2]`,... `z[lg(z)-1]` are `GEN`s containing additional data, depending on the error type.

4.5.19 Type `t_CLOSURE` (closure). This type holds GP functions and closures, in compiled form. The internal detail of this type is subject to change each time the GP language evolves. Hence we do not describe it here and refer to the Developer's Guide. However functions to create or to evaluate `t_CLOSURE`s are documented in Section 12.1.

`long closure_arity(GEN C)` returns the arity of the `t_CLOSURE`.

`long closure_is_variadic(GEN C)` returns 1 if the closure `C` is variadic, 0 else.

4.5.20 Type `t_INFINITY` (infinity).

This type has a single `t_INT` component, which is either 1 or -1, corresponding to $+\infty$ and $-\infty$ respectively.

`GEN mkmoo()` returns $-\infty$

`GEN mkoo()` returns ∞

`long inf_get_sign(GEN x)` returns 1 if x is $+\infty$, and -1 if x is $-\infty$.

4.5.21 Type `t_LIST` (list). this type was introduced for specific `gp` use and is rather inefficient compared to a straightforward linked list implementation (it requires more memory, as well as many unnecessary copies). Hence we do not describe it here and refer to the Developer's Guide.

Implementation note. For the types including an exponent (or a valuation), we actually store a biased nonnegative exponent (bit-ORing the biased exponent to the codeword), obtained by adding a constant to the true exponent: either `HIGHEXPOBIT` (for `t_REAL`) or `HIGHVALPBIT` (for `t_PADIC` and `t_SER`). Of course, this is encapsulated by the exponent/valuation-handling macros and needs not concern the library user.

4.6 PARI variables.

4.6.1 Multivariate objects.

We now consider variables and formal computations. As we have seen in Section 4.5, the codewords for types `t_POL` and `t_SER` encode a "variable number". This is an integer, ranging from 0 to `MAXVARN`. Relative priorities may be ascertained using

```
int varncmp(long v, long w)
```

which is > 0, $= 0$, < 0 whenever v has lower, resp. same, resp. higher priority than w.

The way an object is considered in formal computations depends entirely on its "principal variable number" which is given by the function

```
long gvar(GEN z)
```

which returns a variable number for `z`, even if `z` is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE` which has lower priority than any valid variable number. The variable number of a recursive type which is not a polynomial or power series is the variable number with highest priority among its components. But for polynomials and power series only the "outermost" number counts (we directly access `varn`(x) in the codewords): the representation is not symmetrical at all.

Under `gp`, one needs not worry too much since the interpreter defines the variables as it sees them* and do the right thing with the polynomials produced.

But in library mode, they are tricky objects if you intend to build polynomials yourself (and not just let PARI functions produce them, which is less efficient). For instance, it does not make sense to have a variable number occur in the components of a polynomial whose main variable has a lower priority, even though PARI cannot prevent you from doing it.

4.6.2 Creating variables. A basic difficulty is to "create" a variable. Some initializations are needed before you can use a given integer v as a variable number.

Initially, this is done for 0 and 1 (the variables `x` and `y` under `gp`), and $2, \ldots, 9$ (printed as `t2`, ...`t9`), with decreasing priority.

* The first time a given identifier is read by the GP parser a new variable is created, and it is assigned a strictly lower priority than any variable in use at this point. On startup, before any user input has taken place, 'x' is defined in this way and has initially maximal priority (and variable number 0).

4.6.2.1 User variables. When the program starts, `x` (number 0) and `y` (number 1) are the only available variables, numbers 2 to 9 (decreasing priority) are reserved for building polynomials with predictable priorities.

To define further ones, you may use

`GEN varhigher(const char *s)`

`GEN varlower(const char *s)`

to recover a monomial of degree 1 in a new variable, which is guaranteed to have higer (resp. lower) priority than all existing ones at the time of the function call. The variable is printed as s, but is not part of GP's interpreter: it is not a symbol bound to a value.

On the other hand

`long fetch_user_var(char *s)`: inspects the user variable whose name is the string pointed to by `s`, creating it if needed, and returns its variable number.

```
long v = fetch_user_var("y");
GEN gy = pol_x(v);
```

The function raises an exception if the name is already in use for an `install`ed or built-in function, or an alias. This function is mostly useless since it returns a variable with unpredictable priority. Don't use it to create new variables.

Caveat. You can use `gp_read_str` (see Section 4.7.1) to execute a GP command and create GP variables on the fly as needed:

```
GEN gy = gp_read_str("'y"); /* returns pol_x(v), for some v */
long v = varn(gy);
```

But please note the quote `'y` in the above. Using `gp_read_str("y")` might work, but is dangerous, especially when programming functions to be used under `gp`. The latter reads the value of `y`, as *currently* known by the `gp` interpreter, possibly creating it in the process. But if `y` has been modified by previous `gp` commands (e.g. `y = 1`), then the value of `gy` is not what you expected it to be and corresponds instead to the current value of the `gp` variable (e.g. `gen_1`).

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered v. Returns `NULL` if that variable number is unknown to the interpreter, e.g. it is a user variable. Note that this may not be the same as `pol_x(v)` if assignments have been performed in the interpreter.

4.6.2.2 Temporary variables. You can create temporary variables using

`long fetch_var()` returns a new variable with *lower* priority than any variable currently in use.

`long fetch_var_higher()` returns a new variable with *higher* priority than any variable currently in use.

After the statement `v = fetch_var()`, you can use `pol_1(v)` and `pol_x(v)`. The variables created in this way have no identifier assigned to them though, and are printed as `tnumber`. You can assign a name to a temporary variable, after creating it, by calling the function

`void name_var(long n, char *s)`

after which the output machinery will use the name `s` to represent the variable number `n`. The GP parser will *not* recognize it by that name, however, and calling this on a variable known to `gp` raises an error. Temporary variables are meant to be used as free variables to build polynomials and power series, and you should never assign values or functions to them as you would do with variables under `gp`. For that, you need a user variable.

All objects created by `fetch_var` are on the heap and not on the stack, thus they are not subject to standard garbage collecting (they are not destroyed by a `gerepile` or `set_avma(ltop)` statement). When you do not need a variable number anymore, you can delete it using

`long delete_var()`

which deletes the *latest* temporary variable created and returns the variable number of the previous one (or simply returns 0 if none remain). Of course you should make sure that the deleted variable does not appear anywhere in the objects you use later on. Here is an example:

```
long first = fetch_var();
long n1 = fetch_var();
long n2 = fetch_var(); /* prepare three variables for internal use */
...
/* delete all variables before leaving */
do { num = delete_var(); } while (num && num <= first);
```

The (dangerous) statement

```
while (delete_var()) /* empty */;
```

removes all temporary variables in use.

4.6.3 Comparing variables.

Let us go back to `varncmp`. There is an interesting corner case, when one of the compared variables (from `gvar`, say) is `NO_VARIABLE`. In this case, `varncmp` declares it has lower priority than any other variable; of course, comparing `NO_VARIABLE` with itself yields 0 (same priority);

In addition to `varncmp` we have

`long varnmax(long v, long w)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the highest priority. This function always returns a valid variable number unless it is comparing `NO_VARIABLE` to itself.

`long varnmin(long x, long y)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the lowest priority. Note that when comparing a true variable with `NO_VARIABLE`, this function returns `NO_VARIABLE`, which is not a valid variable number.

4.7 Input and output.

Two important aspects have not yet been explained which are specific to library mode: input and output of PARI objects.

4.7.1 Input.

For input, PARI provides several powerful high level functions which enable you to input your objects as if you were under `gp`. In fact, it *is* essentially the GP syntactical parser.

There are two similar functions available to parse a string:

`GEN gp_read_str(const char *s)`

`GEN gp_read_str_multiline(const char *s, char *last)`

Both functions read the whole string `s`. The function `gp_read_str` ignores newlines: it assumes that the input is one expression and returns the result of this expression.

The function `gp_read_str_multiline` processes the text in the same way as the GP command `read`: newlines are significant and can be used to separate expressions. The return value is that of the last nonempty expression evaluated.

In `gp_read_str_multiline`, if `last` is not `NULL`, then `*last` receives the last character from the *filtered* input: this can be used to check if the last character was a semi-colon (to hide the output in interactive usage). If (and only if) the input contains no statements, then `*last` is set to `0`.

For both functions, `gp`'s metacommands *are* recognized.

Two variants allow to specify a default precision while evaluating the string:

`GEN gp_read_str_prec(const char *s, long prec)` As `gp_read_str`, but set the precision to `prec` words while evaluating s.

`GEN gp_read_str_bitprec(const char *s, long bitprec)` As `gp_read_str`, but set the precision to `bitprec` bits while evaluating s.

Note. The obsolete form

`GEN readseq(char *t)`

still exists for backward compatibility (assumes filtered input, without spaces or comments). Don't use it.

To read a `GEN` from a file, you can use the simpler interface

`GEN gp_read_stream(FILE *file)`

which reads a character string of arbitrary length from the stream `file` (up to the first complete expression sequence), applies `gp_read_str` to it, and returns the resulting `GEN`. This way, you do not have to worry about allocating buffers to hold the string. To interactively input an expression, use `gp_read_stream(stdin)`. Return `NULL` when there are no more expressions to read (we reached EOF).

Finally, you can read in a whole file, as in GP's `read` statement

`GEN gp_read_file(char *name)`

As usual, the return value is that of the last nonempty expression evaluated. There is one technical exception: if `name` is a *binary* file (from `writebin`) containing more than one object, a `t_VEC` containing them all is returned. This is because binary objects bypass the parser, hence reading them has no useful side effect.

4.7.2 Output to screen or file, output to string.

General output functions return nothing but print a character string as a side effect. Low level routines are available to write on PARI output stream `pari_outfile` (`stdout` by default):

`void pari_putc(char c)`: write character `c` to the output stream.

`void pari_puts(char *s)`: write `s` to the output stream.

`void pari_flush()`: flush output stream; most streams are buffered by default, this command makes sure that all characters output so are actually written.

`void pari_printf(const char *fmt, ...)`: the most versatile such function. `fmt` is a character string similar to the one `printf` uses. In there, `%` characters have a special meaning, and describe how to print the remaining operands. In addition to the standard format types (see the GP function `printf`), you can use the *length modifier* `P` (for PARI of course!) to specify that an argument is a `GEN`. For instance, the following are valid conversions for a `GEN` argument

`%Ps`	*convert to* `char*` *(will print an arbitrary* `GEN`*)*
`%P.10s`	*convert to* `char*`*, truncated to 10 chars*
`%P.2f`	*convert to floating point format with 2 decimals*
`%P4d`	*convert to integer, field width at least 4*

```
pari_printf("x[%d] = %Ps is not invertible!\n", i, gel(x,i));
```

Here `i` is an `int`, `x` a `GEN` which is not a leaf (presumably a vector, or a polynomial) and this would insert the value of its i-th `GEN` component: `gel(x,i)`.

Simple but useful variants to `pari_printf` are

`void output(GEN x)` prints `x` in raw format, followed by a newline and a buffer flush. This is more or less equivalent to

```
pari_printf("%Ps\n", x);
pari_flush();
```

`void outmat(GEN x)` as above except if x is a `t_MAT`, in which case a multi-line display is used to display the matrix. This is prettier for small dimensions, but quickly becomes unreadable and cannot be pasted and reused for input. If all entries of x are small integers, you may use the recursive features of `%Pd` and obtain the same (or better) effect with

```
pari_printf("%Pd\n", x);
pari_flush();
```

A variant like `"%5Pd"` would improve alignment by imposing 5 chars for each coefficient. Similarly if all entries are to be converted to floats, a format like `"%5.1Pf"` could be useful.

These functions write on (PARI's idea of) standard output, and must be used if you want your functions to interact nicely with `gp`. In most programs, this is not a concern and it is more flexible to write to an explicit `FILE*`, or to recover a character string:

`void pari_fprintf(FILE *file, const char *fmt, ...)` writes the remaining arguments to stream `file` according to the format specification `fmt`.

`char* pari_sprintf(const char *fmt, ...)` produces a string from the remaining arguments, according to the PARI format `fmt` (see `printf`). This is the `libpari` equivalent of `strprintf`, and returns a `malloc`'ed string, which must be freed by the caller. Note that contrary to the analogous `sprintf` in the `libc` you do not provide a buffer (leading to all kinds of buffer overflow concerns); the function provided is actually closer to the GNU extension `asprintf`, although the latter has a different interface.

Simple variants of `pari_sprintf` convert a `GEN` to a `malloc`'ed ASCII string, which you must still `free` after use:

`char* GENtostr(GEN x)`, using the current default output format (`prettymat` by default).

`char* GENtoTeXstr(GEN x)`, suitable for inclusion in a TeX file.

Note that we have `va_list` analogs of the functions of `printf` type seen so far:

`void pari_vprintf(const char *fmt, va_list ap)`

`void pari_vfprintf(FILE *file, const char *fmt, va_list ap)`

`char* pari_vsprintf(const char *fmt, va_list ap)`

4.7.3 Errors.

If you want your functions to issue error messages, you can use the general error handling routine `pari_err`. The basic syntax is

```
pari_err(e_MISC, "error message");
```

This prints the corresponding error message and exit the program (in library mode; go back to the `gp` prompt otherwise). You can also use it in the more versatile guise

```
pari_err(e_MISC, format, ...);
```

where `format` describes the format to use to write the remaining operands, as in the `pari_printf` function. For instance:

```
pari_err(e_MISC, "x[%d] = %Ps is not invertible!", i, gel(x,i));
```

The simple syntax seen above is just a special case with a constant format and no remaining arguments. The general syntax is

`void pari_err(numerr, ...)`

where `numerr` is a codeword which specifies the error class and what to do with the remaining arguments and what message to print. For instance, if x is a `GEN` with internal type `t_STR`, say, `pari_err(e_TYPE,"extgcd", `x`)` prints the message:

```
  ***   incorrect type in extgcd (t_STR),
```

See Section 11.4 for details. In the libpari code itself, the general-purpose `e_MISC` is used sparingly: it is so flexible that the corresponding error contexts (`t_ERROR`) become hard to use reliably. Other more rigid error types are generally more useful: for instance the error context attached to the `e_TYPE` exception above is precisely documented and contains `"extgcd"` and x (not only its type) as readily available components.

4.7.4 Warnings.

To issue a warning, use

`void pari_warn(warnerr, ...)` In that case, of course, we do *not* abort the computation, just print the requested message and go on. The basic example is

```
pari_warn(warner, "Strategy 1 failed. Trying strategy 2")
```

which is the exact equivalent of `pari_err(e_MISC,...)` except that you certainly do not want to stop the program at this point, just inform the user that something important has occurred; in particular, this output would be suitably highlighted under `gp`, whereas a simple `printf` would not.

The valid *warning* keywords are `warner` (general), `warnprec` (increasing precision), `warnmem` (garbage collecting) and `warnfile` (error in file operation), used as follows:

```
pari_warn(warnprec, "bnfinit", newprec);
pari_warn(warnmem,  "bnfinit");
pari_warn(warnfile, "close", "afile");  /* error when closing "afile" */
```

4.7.5 Debugging output.

For debugging output, you can use the standard output functions, `output` and `pari_printf` mainly. Corresponding to the `gp` metacommand `\x`, you can also output the hexadecimal tree attached to an object:

`void dbgGEN(GEN x, long nb = -1)`, displays the recursive structure of `x`. If `nb` $= -1$, the full structure is printed, otherwise the leaves (nonrecursive components) are truncated to `nb` words.

The function `output` is vital under debuggers, since none of them knows how to print PARI objects by default. Seasoned PARI developers add the following `gdb` macro to their `.gdbinit`:

```
define oo
  call output((GEN)$arg0)
end
define xx
  call dbgGEN($arg0,-1)
end
```

Typing `i x` at a breakpoint in `gdb` then prints the value of the `GEN x` (provided the optimizer has not put it into a register, but it is rarely a good idea to debug optimized code).

The global variables **DEBUGLEVEL** and **DEBUGMEM** (corresponding to the default **debug** and **debugmem**) are used throughout the PARI code to govern the amount of diagnostic and debugging output, depending on their values. You can use them to debug your own functions, especially if you `install` the latter under `gp`. Note that `DEBUGLEVEL` is redefined in each code module, attaching it to a particular debug domain (see `setdebug`).

`void setalldebug(long L)` sels all `DEBUGLEVEL` incarnations (all debug domains) to L.

`void dbg_pari_heap(void)` print debugging statements about the PARI stack, heap, and number of variables used. Corresponds to `\s` under gp.

4.7.6 Timers and timing output.

To handle timings in a reentrant way, PARI defines a dedicated data type, `pari_timer`, together with the following methods:

`void timer_start(pari_timer *T)` start (or reset) a timer.

`long timer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Resets the timer as a side effect. Assume T was started by `timer_start`.

`long timer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer. Assume T was started by `timer_start`.

`void walltimer_start(pari_timer *T)` start a timer, as if it had been started at the Unix epoch (see `getwalltime`).

`long walltimer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last checked. Assume T was started by `walltimer_start`.

`long walltimer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer. Assume T was started by `walltimer_start`.

`long timer_printf(pari_timer *T, char *format, ...)` This diagnostics function is equivalent to the following code

```
err_printf("Time ")
... prints remaining arguments according to format ...
err_printf(": %ld", timer_delay(T));
```

Resets the timer as a side effect.

They are used as follows:

```
pari_timer T;
timer_start(&T); /* initialize timer */
...
printf("Total time: %ldms\n", timer_delay(&T));
```

or

```
pari_timer T;
timer_start(&T);
for (i = 1; i < 10; i++) {
  ...
  timer_printf(&T, "for i = %ld (L[i] = %Ps)", i, gel(L,i));
}
```

The following functions provided the same functionality, in a nonreentrant way, and are now deprecated.

`long timer(void)`

`long timer2(void)`

`void msgtimer(const char *format, ...)`

The following function implements `gp`'s timer and should not be used in libpari programs: `long gettime(void)` equivalent to `timer_delay(`T`)` attached to a private timer T.

4.8 Iterators, Numerical integration, Sums, Products.

4.8.1 Iterators. Since it is easier to program directly simple loops in library mode, some GP iterators are mainly useful for GP programming. Here are the others:

• `fordiv` is a trivial iteration over a list produced by `divisors`.

• `forell`, `forqfvec` and `forsubgroup` are currently not implemented as an iterator but as a procedure with callbacks.

`void forell(void *E, long fun(void*, GEN), GEN a, GEN b, long flag)` goes through the same curves as `forell(ell,a,b,,flag)`, calling `fun(E, ell)` for each curve `ell`, stopping if `fun` returns a nonzero value.

`void forqfvec(void *E, long (*fun)(void *, GEN, GEN, double), GEN q, GEN b)` : Evaluate `fun(E,U,v,m)` on all v such that $q(Uv) < b$, where U is a `t_MAT`, v is a `t_VECSMALL` and $m = q(v)$ is a C double. The function `fun` must return 0, unless `forqfvec` should stop, in which case, it should return 1.

`void forqfvec1(void *E, long (*fun)(void *, GEN), GEN q, GEN b)`: Evaluate `fun(E,v)` on all v such that $q(v) < b$, where v is a `t_COL`. The function `fun` must return 0, unless `forqfvec` should stop, in which case, it should return 1.

`void forsubgroup(void *E, long fun(void*, GEN), GEN G, GEN B)` goes through the same subgroups as `forsubgroup(H = G, B,)`, calling `fun(E, H)` for each subgroup H, stopping if `fun` returns a nonzero value.

• `forprime` and `forprimestep`, iterators over primes and primes in arithmetic progressions, for which we refer you to the next subsection.

• `forcomposite`, we provide an iterator over composite integers:

`int forcomposite_init(forcomposite_t *T, GEN a, GEN b)` initialize an iterator T over composite integers in $[a, b]$; over composites $\geq a$ if $b =$ `NULL`. We must have $a \geq 0$. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise.

`GEN forcomposite_next(forcomposite_t *T)` returns the next composite in the range, assuming that T was initialized by `forcomposite_init`.

• `forvec`, for which we provide a convenient iterator. To initialize the analog of `forvec(X = v, ..., flag)`, call

`int forvec_init(forvec_t *T, GEN v, long flag)` initialize an iterator T over the vectors generated by `forvec(X =` v`,..., flag)`. This returns 0 if this vector list is empty, and 1 otherwise.

`GEN forvec_next(forvec_t *T)` returns the next element in the `forvec` sequence, or `NULL` if we are done. The return value must be used immediately or copied since the next call to the iterator destroys it: the relevant vector is updated in place. The iterator works hard to not use up PARI stack, and is more efficient when all lower bounds in the initialization vector v are integers. In that case, the cost is linear in the number of tuples enumerated, and you can expect to run over more than 10^9 tuples per minute. If speed is critical and all integers involved would fit in C `long`s, write a simple direct backtracking algorithm yourself.

• `forpart` is a variant of `forvec` which iterates over partitions. See the documentation of the `forpart` GP function for details. This function is available as a loop with callbacks:

`void forpart(void *data, long (*call)(void*, GEN), long k, GEN a, GEN n)`

It is also available as an iterator:

`void forpart_init(forpart_t *T, long k, GEN a, GEN n)` initializes an iterator over the partitions of k, with length restricted by n, and components restricted by a, either of which can be set to `NULL` to run without restriction.

`GEN forpart_next(forpart_t *T)` returns the next partition, or `NULL` when all partitions have been exhausted.

`GEN forpart_prev(forpart_t *T)` returns the previous partition, or `NULL` when all partitions have been exhausted.

In both cases, the partition must be used or copied before the next call since it is returned from a state array which will be modified in place. You may *not* mix calls to `forpart_next` and `forpart_prev`: the first one called determines the ordering used to iterate over the partitions; you can not go back since the `forpart_t` structure is used in incompatible ways.

• `forperm` to loop over permutations of k. See the documentation of the `forperm` GP function for details. This function is available as an iterator:

`void forperm_init(forperm_t *T, GEN k)` initializes an iterator over the permutations of k (`t_INT`, `t_VEC` or `t_VECSMALL`).

`GEN forperm_next(forperm_t *T)` returns the next permutation as a `t_VECSMALL` or `NULL` whell all permutations have been exhausted. The permutation must be used or copied before the next call since it is returned from a state array which will be modified in place.

• `forsubset` to loop over subsets. See the documentation of the `forsubset` GP function for details. This function is available as two iterators:

`void forallsubset_init(forsubset_t *T, long n)`

`void forksubset_init(forsubset_t *T, long n, long k)`

It is also available in generic form:

`void forsubset_init(forsubset_t *T, GEN nk)` where `nk` is either a `t_INT` n or a `t_VEC` with two integral components $[n, k]$.

In all three cases, `GEN forsubset_next(forsubset_t *T)` returns the next subset as a `t_VECSMALL` or `NULL` when all subsets have been exhausted.

4.8.2 Iterating over primes.

The library provides a high-level iterator, which stores its (private) data in a `struct forprime_t` and runs over arbitrary ranges of primes, without ever overflowing.

The iterator has two flavors, one providing the successive primes as `ulong`s, the other as `GEN`. They are initialized as follows, where we expect to run over primes $\geq a$ and $\leq b$:

`int u_forprime_init(forprime_t *T, ulong a, ulong b)` for the `ulong` variant, where $b =$ `ULONG_MAX` means we will run through all primes representable in a `ulong` type.

`int forprime_init(forprime_t *T, GEN a, GEN b)` for the `GEN` variant, where $b =$ `NULL` means $+\infty$.

`int forprimestep_init(forprime_t *T, GEN a, GEN b, GEN q)` initialize an iterator T over primes in an arithmetic progression, $p \geq a$ and $p \leq b$ (where $b =$ NULL means $+\infty$). The argument q is either a t_INT ($p \equiv a \pmod{q}$) or a t_INTMOD Mod(c,N) and we restrict to that congruence class.

All variants return 1 on success, and 0 if the iterator would run over an empty interval (if $a > b$, for instance). They allocate the forprime_t data structure on the PARI stack.

The successive primes are then obtained using

`GEN forprime_next(forprime_t *T)`, returns NULL if no more primes are available in the interval and the next suitable prime as a t_INT otherwise.

`ulong u_forprime_next(forprime_t *T)`, returns 0 if no more primes are available in the interval and fitting in an ulong and the next suitable prime otherwise.

These two functions leave alone the PARI stack, and write their state information in the preallocated forprime_t struct. The typical usage is thus:

```
forprime_t T;
GEN p;
pari_sp av = avma, av2;
forprime_init(&T, gen_2, stoi(1000));
av2 = avma;
while ( (p = forprime_next(&T)) )
{
  ...
  if ( prime_is_OK(p) ) break;
  set_avma(av2); /* delete garbage accumulated in this iteration */
}
set_avma(av); /* delete all */
```

Of course, the final set_avma(av) could be replaced by a gerepile call. Beware that swapping the av2 = avma and forprime_init call would be incorrect: the first set_avma(av2) would delete the forprime_t structure!

4.8.3 Parallel iterators.

Theses iterators loops over the values of a t_CLOSURE taken at some data, where the evaluations are done in parallel.

- parfor. To initialize the analog of parfor(i = a, b, ...), call

`void parfor_init(parfor_t *T, GEN a, GEN b, GEN code)` initialize an iterator over the evaluation of code on the integers between a and b.

`GEN parfor_next(parfor_t *T)` returns a t_VEC [i,code(i)] where i is one of the integers and code(i) is the evaluation, NULL when all data have been exhausted. Once it happens, parfor_next must not be called anymore with the same initialization.

`void parfor_stop(parfor_t *T)` needs to be called when leaving the iterator before parfor_next returned NULL.

The following returns an integer $1 \leq i \leq N$ such that fun(i) is not zero, or NULL.

```
GEN
```

```
parfirst(GEN fun, GEN N)
{
  parfor_t T;
  GEN e;
  parfor_init(&T, gen_1, N, fun);
  while ((e = parfor_next(&T)))
  {
    GEN i = gel(e,1), funi = gel(e,2);
    if (!gequal0(funi))
    { /* found: stop the iterator and return the index */
      parfor_stop(&T);
      return i;
    }
  }
  return NULL; /* not found */
}
```

• `parforeach`. To initialize the analog of `parforeach(V, X, ...)`, call

`void parforeach_init(parforeach_t *T, GEN V, GEN code)` initialize an iterator over the evaluation of `code` on the components of V.

`GEN parforeach_next(parforeach_t *T)` returns a `t_VEC [V[i],code(V[i])]` where $V[i]$ is one of the components of V and `code(V[i])` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforprime_next` must not be called anymore with the same initialization.

`void parforeach_stop(parforeach_t *T)` needs to be called when leaving the iterator before `parforeach_next` returned `NULL`.

• `parforprime`. To initialize the analog of `parforprime(p = a, b, ...)`, call

`void parforprime_init(parforprime_t *T, GEN a, GEN b, GEN code)` initialize an iterator over the evaluation of `code` on the prime numbers between a and b.

• `parforprimestep`. To initialize the analog of `parforprimestep(p = a, b, q, ...)`, call

`void parforprimestep_init(parforprime_t *T, GEN a, GEN b, GEN q, GEN code)` initialize an iterator over the evaluation of `code` on the prime numbers between a and b in the congruence class defined by q.

`GEN parforprime_next(parforprime_t *T)` returns a `t_VEC [p,code(p)]` where p is one of the prime numbers and `code(p)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforprime_next` must not be called anymore with the same initialization.

`void parforprime_stop(parforprime_t *T)` needs to be called when leaving the iterator before `parforprime_next` returned `NULL`.

• `parforvec`. To initialize the analog of `parforvec(X = V, ..., flag)`, call

`void parforvec_init(parforvec_t *T, GEN V, GEN code, long flag)` initialize an iterator over the evaluation of `code` on the vectors specified by `V` and `flag`, see `forvec` for detail.

`GEN parforvec_next(parforvec_t *T)` returns a `t_VEC [v,code(v)]` where v is one of the vectors and `code(v)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforvec_next` must not be called anymore with the same initialization.

`void parforvec_stop(parforvec_t *T)` needs to be called when leaving the iterator before `parforvec_next` returned `NULL`.

4.8.4 Numerical analysis.

Numerical routines code a function (to be integrated, summed, zeroed, etc.) with two parameters named

```
void *E;
GEN (*eval)(void*, GEN)
```

The second is meant to contain all auxiliary data needed by your function. The first is such that `eval(x, E)` returns your function evaluated at `x`. For instance, one may code the family of functions $f_t : x \to (x + t)^2$ via

```
GEN fun(void *t, GEN x) { return gsqr(gadd(x, (GEN)t)); }
```

One can then integrate f_1 between a and b with the call

```
intnum((void*)stoi(1), &fun, a, b, NULL, prec);
```

Since you can set `E` to a pointer to any `struct` (typecast to `void*`) the above mechanism handles arbitrary functions. For simple functions without extra parameters, you may set `E = NULL` and ignore that argument in your function definition.

4.9 Catching exceptions.

4.9.1 Basic use.

PARI provides a mechanism to trap exceptions generated via `pari_err` using the `pari_CATCH` construction. The basic usage is as follows

```
pari_CATCH(err_code) {
   recovery branch
}
pari_TRY {
   main branch
}
pari_ENDCATCH
```

This fragment executes the main branch, then the recovery branch *if* exception `err_code` is thrown, e.g. `e_TYPE`. See Section 11.4 for the description of all error classes. The special error code `CATCH_ALL` is available to catch all errors.

One can replace the `pari_TRY` keyword by `pari_RETRY`, in which case once the recovery branch is run, we run the main branch again, still catching the same exceptions.

Restrictions.

• Such constructs can be nested without adverse effect, the innermost handler catching the exception.

• It is *valid* to leave either branch using `pari_err`.

• It is *invalid* to use C flow control instructions (`break`, `continue`, `return`) to directly leave either branch without seeing the `pari_ENDCATCH` keyword. This would leave an invalid structure in the exception handler stack, and the next exception would crash.

• In order to leave using `break`, `continue` or `return`, one must precede the keyword by a call to

`void pari_CATCH_reset()` disable the current handler, allowing to leave without adverse effect.

4.9.2 Advanced use.

In the recovery branch, the exception context can be examined via the following helper routines:

`GEN pari_err_last()` returns the exception context, as a `t_ERROR`. The exception E returned by `pari_err_last` can be rethrown, using

```
pari_err(0, E);
```

`long err_get_num(GEN E)` returns the error symbolic name. E.g `e_TYPE`.

`GEN err_get_compo(GEN E, long i)` error i-th component, as documented in Section 11.4.

For instance

```
pari_CATCH(CATCH_ALL) { /* catch everything */
   GEN x, E = pari_err_last();
   long code = err_get_num(E);
   if (code != e_INV) pari_err(0, E); /* unexpected error, rethrow */
   x = err_get_compo(E, 2);
   /* e_INV has two components, 1: function name 2: noninvertible x */
   if (typ(x) != t_INTMOD) pari_err(0, E); /* unexpected type, rethrow */
   pari_CATCH_reset();
   return x; /* leave ! */
    ...
} pari_TRY {
   main branch
}
pari_ENDCATCH
```

4.10 A complete program.

Now that the preliminaries are out of the way, the best way to learn how to use the library mode is to study a detailed example. We want to write a program which computes the gcd of two integers, together with the Bezout coefficients. We shall use the standard quadratic algorithm which is not optimal but is not too far from the one used in the PARI function **bezout**.

Let x, y two integers and initially $\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that

$$\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To apply the ordinary Euclidean algorithm to the right hand side, multiply the system from the left by $\begin{pmatrix} 0 & 1 \\ 1 & -q \end{pmatrix}$, with $q = \texttt{floor}(x/y)$. Iterate until $y = 0$ in the right hand side, then the first line of the system reads

$$s_x x + s_y y = \gcd(x, y).$$

In practice, there is no need to update s_y and t_y since $\gcd(x, y)$ and s_x are enough to recover s_y. The following program is now straightforward. A couple of new functions appear in there, whose description can be found in the technical reference manual in Chapter 5, but whose meaning should be clear from their name and the context.

This program can be found in `examples/extgcd.c` together with a proper `Makefile`. You may ignore the first comment

```
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
```

which instruments the program so that `gp2c-run extgcd.c` can import the `extgcd()` routine into an instance of the `gp` interpreter (under the name `gcdex`). See the `gp2c` manual for details.

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
  pari_sp av = avma;
  GEN ux = gen_1, vx = gen_0, a = A, b = B;
  if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
  if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
  if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;
    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
  }
  *U = ux;
  *V = diviiexact( subii(a, mulii(A,ux)), B );
  gerepileall(av, 3, &a, U, V); return a;
}
int
main()
{
  GEN x, y, d, u, v;
  pari_init(1000000,2);
  printf("x = "); x = gp_read_stream(stdin);
  printf("y = "); y = gp_read_stream(stdin);
  d = extgcd(x, y, &u, &v);
  pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
  pari_close();
  return 0;
}
```

For simplicity, the inner loop does not include any garbage collection, hence memory use is quadratic in the size of the inputs instead of linear. Here is a better version of that loop:

```
  pari_sp av = avma;
  ...
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;
    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
    if (gc_needed(av,1))
      gerepileall(av, 4, &a, &b, &ux, &vx);
```

```
}
```

Chapter 5: Technical Reference Guide: the basics

In the following chapters, we describe all public low-level functions of the PARI library. These include specialized functions for handling all the PARI types. Simple higher level functions, such as arithmetic or transcendental functions, are described in Chapter 3 of the GP user's manual; we will eventually see more general or flexible versions in the chapters to come. A general introduction to the major concepts of PARI programming can be found in Chapter 4, which you should really read first.

We shall now study specialized functions, more efficient than the library wrappers, but sloppier on argument checking and damage control; besides speed, their main advantage is to give finer control about the inner workings of generic routines, offering more options to the programmer.

Important advice. Generic routines eventually call lower level functions. Optimize your algorithms first, not overhead and conversion costs between PARI routines. For generic operations, use generic routines first; do not waste time looking for the most specialized one available unless you identify a genuine bottleneck, or you need some special behavior the generic routine does not offer. The PARI source code is part of the documentation; look for inspiration there.

The type `long` denotes a `BITS_IN_LONG`-bit signed long integer (32 or 64 bits). The type `ulong` is defined as `unsigned long`. The word *stack* always refer to the PARI stack, allocated through an initial `pari_init` call. Refer to Chapters 1–2 and 4 for general background.

We shall often refer to the notion of *shallow* function, which means that some components of the result may point to components of the input, which is more efficient than a *deep* copy (full recursive copy of the object tree). Such outputs are not suitable for `gerepileupto` and particular care must be taken when garbage collecting objects which have been input to shallow functions: corresponding outputs also become invalid and should no longer be accessed.

A function is *not stack clean* if it leaves intermediate data on the stack besides its output, for efficiency reasons.

5.1 Initializing the library.

The following functions enable you to start using the PARI functions in a program, and cleanup without exiting the whole program.

5.1.1 General purpose.

`void pari_init(size_t size, ulong maxprime)` initialize the library, with a stack of `size` bytes and a prime table up to the maximum of `maxprime` and 2^{16}. Unless otherwise mentioned, no PARI function will function properly before such an initialization.

`void pari_close(void)` stop using the library (assuming it was initialized with `pari_init`) and frees all allocated objects.

5.1.2 Technical functions.

`void pari_init_opts(size_t size, ulong maxprime, ulong opts)` as `pari_init`, more flexible. `opts` is a mask of flags among the following:

`INIT_JMPm`: install PARI error handler. When an exception is raised, the program is terminated with `exit(1)`.

`INIT_SIGm`: install PARI signal handler.

`INIT_DFTm`: initialize the `GP_DATA` environment structure. This one *must* be enabled once. If you close pari, then restart it, you need not reinitialize `GP_DATA`; if you do not, then old values are restored.

`INIT_noPRIMEm`: do not compute the prime table (ignore the `maxprime` argument). The user *must* call `pari_init_primes` later.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_init` to initialize the multi-thread engine. If this flag is set, `pari_mt_init()` will need to be called manually. See `examples/pari-mt.c` for an example.

`INIT_noINTGMPm`: do not install PARI-specific GMP memory functions. This option is ignored when the GMP library is not in use. You may install PARI-specific GMP memory functions later by calling

`void pari_kernel_init(void)`

and restore the previous values using

`void pari_kernel_close(void)`

This option should not be used without a thorough understanding of the problem you are trying to solve. The GMP memory functions are global variables used by the GMP library. If your program is linked with two libraries that require these variables to be set to different values, conflict ensues. To avoid a conflict, the proper solution is to record their values with `mp_get_memory_functions` and to call `mp_set_memory_functions` to restore the expected values each time the code switches from using one library to the other. Here is an example:

```
void *(*pari_alloc_ptr) (size_t);
void *(*pari_realloc_ptr) (void *, size_t, size_t);
void (*pari_free_ptr) (void *, size_t);
void *(*otherlib_alloc_ptr) (size_t);
void *(*otherlib_realloc_ptr) (void *, size_t, size_t);
void (*otherlib_free_ptr) (void *, size_t);

void init(void)
{
  pari_init(8000000, 500000);
  mp_get_memory_functions(&pari_alloc_ptr,&pari_realloc_ptr,
                          &pari_free_ptr);
  otherlib_init();
  mp_get_memory_functions(&otherlib_alloc_ptr,&otherlib_realloc_ptr,
                          &otherlib_free_ptr);
}
void function_that_use_pari(void)
{
```

```
  mp_set_memory_functions(pari_alloc_ptr,pari_realloc_ptr,
                          pari_free_ptr);
  /*use PARI functions*/
}
void function_that_use_otherlib(void)
{
  mp_set_memory_functions(otherlib_alloc_ptr,otherlib_realloc_ptr,
                          otherlib_free_ptr);
  /*use OTHERLIB functions*/
}
```

`void pari_close_opts(ulong init_opts)` as `pari_close`, for a library initialized with a mask of options using `pari_init_opts`. `opts` is a mask of flags among

`INIT_SIGm`: restore `SIG_DFL` default action for signals tampered with by PARI signal handler.

`INIT_DFTm`: frees the `GP_DATA` environment structure.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_close` to close the multi-thread engine. `INIT_noINTGMPm`: do not restore GMP memory functions.

`void pari_sig_init(void (*f)(int))` install the signal handler `f` (see `signal(2)`): the signals `SIGBUS`, `SIGFPE`, `SIGINT`, `SIGBREAK`, `SIGPIPE` and `SIGSEGV` are concerned.

`void pari_init_primes(ulong maxprime)` Initialize the PARI primes. This function is called by `pari_init(...,maxprime)`. It is provided for users calling `pari_init_opts` with the flag `INIT_noPRIMEm`.

`void pari_sighandler(int signum)` the actual signal handler that PARI uses. This can be used as argument to `pari_sig_init` or `signal(2)`.

`void pari_stackcheck_init(void *stackbase)` controls the system stack exhaustion checking code in the GP interpreter. This should be used when the system stack base address change or when the address seen by `pari_init` is too far from the base address. If `stackbase` is `NULL`, disable the check, else set the base address to `stackbase`. It is normally used this way

```
int thread_start (...)
{
  long first_item_on_the_stack;
  ...
  pari_stackcheck_init(&first_item_on_the_stack);
}
```

`int pari_daemon(void)` forks a PARI daemon, detaching from the main process group. The function returns 1 in the parent, and 0 in the forked son.

`void paristack_setsize(size_t rsize, size_t vsize)` sets the default `parisize` to `rsize` and the default `parisizemax` to `vsize`, and reallocate the stack to match these value, destroying its content. Generally used just after `pari_init`.

`void paristack_resize(ulong newsize)` changes the current stack size to `newsize` (double it if `newsize` is 0). The new size is clipped to be at least the current stack size and at most `parisizemax`. The stack content is not affected by this operation.

void parivstack_reset(void) resets the current stack to its default size parisize. This is used to recover memory after a computation that enlarged the stack. This function destroys the content of the enlarged stack (between the old and the new bottom of the stack). Before calling this function, you must ensure that avma lies within the new smaller stack.

void paristack_newrsize(ulong newsize) *(does not return)*. Library version of

```
default(parisize, "newsize")
```

Set the default parisize to newsize, or double parisize if newsize is equal to 0, then call cb_pari_err_recover(-1).

void parivstack_resize(ulong newsize) *(does not return)*. Library version of

```
default(parisizemax, "newsize")
```

Set the default parisizemax to newsize and call cb_pari_err_recover(-1).

5.1.3 Notions specific to the GP interpreter.

An entree is the generic object attached to an identifier (a name) in GP's interpreter, be it a built-in or user function, or a variable. For a function, it has at least the following fields:

char *name: the name under which the interpreter knows us.

void *value: a pointer to the C function to call.

long menu: a small integer ≥ 1 (to which group of function help do we belong, for the ?n help menu).

char *code: the prototype code.

char *help: the help text for the function.

A routine in GP is described to the analyzer by an entree structure. Built-in PARI routines are grouped in *modules*, which are arrays of entree structs, the last of which satisfy name = NULL (sentinel). There are currently four modules in PARI/GP:

- general functions (functions_basic, known to libpari),
- gp-specific functions (functions_gp),

and two modules of obsolete functions. The function pari_init initializes the interpreter and declares all symbols in functions_basic. You may declare further functions on a case by case basis or as a whole module using

void pari_add_function(entree *ep) adds a single routine to the table of symbols in the interpreter. It assumes pari_init has been called.

void pari_add_module(entree *mod) adds all the routines in module mod to the table of symbols in the interpreter. It assumes pari_init has been called.

For instance, gp implements a number of private routines, which it adds to the default set via the calls

```
pari_add_module(functions_gp);
```

A GP default is likewise attached to a helper routine, that is run when the value is consulted, or changed by default0 or setdefault. Such routines are grouped in the module functions_default.

`void pari_add_defaults_module(entree *mod)` adds all the defaults in module `mod` to the interpreter. It assumes that `pari_init` has been called. From this point on, all defaults in module `mod` are known to `setdefault` and friends.

5.1.4 Public callbacks.

The `gp` calculator associates elaborate functions (for instance the break loop handler) to the following callbacks, and so can you:

`void (*cb_pari_ask_confirm)(const char *s)` initialized to `NULL`. Called with argument s whenever PARI wants confirmation for action s, for instance in `secure` mode.

`void (*cb_pari_init_histfile)(void)` initialized to `NULL`. Called when the `histfile` default is changed. The intent is for that callback to read the file content, append it to history in memory, then dump the expanded history to the new `histfile`.

`int (*cb_pari_is_interactive)(void)`; initialized to `NULL`.

`void (*cb_pari_quit)(long)` initialized to a no-op. Called when `gp` must evaluate the `quit` command.

`void (*cb_pari_start_output)(void)` initialized to `NULL`.

`int (*cb_pari_handle_exception)(long)` initialized to `NULL`. If not `NULL`, this routine is called with argument -1 on `SIGINT`, and argument `err` on error `err`. If it returns a nonzero value, the error or signal handler returns, in effect further ignoring the error or signal, otherwise it raises a fatal error. A possible simple-minded handler, used by the `gp` interpreter, is

`int gp_handle_exception(long err)` if the `breakloop` default is enabled (set to 1) and `cb_pari_break_loop` is not `NULL`, we call this routine with `err` argument and return the result.

`int (*cb_pari_err_handle)(GEN)` If not `NULL`, this routine is called with a `t_ERROR` argument from `pari_err`. If it returns a nonzero value, the error returns, in effect further ignoring the error, otherwise it raises a fatal error.

The default behavior is to print a descriptive error message (display the error), then return 0, thereby raising a fatal error. This differs from `cb_pari_handle_exception` in that the function is not called on `SIGINT` (which do not generate a `t_ERROR`), only from `pari_err`. Use `cb_pari_sigint` if you need to handle `SIGINT` as well.

The following function can be used by `cb_pari_err_handle` to display the error message.

`const char* closure_func_err()` return a statically allocated string holding the name of the function that triggered the error. Return NULL if the error was not caused by a function.

`int (*cb_pari_break_loop)(int)` initialized to `NULL`.

`void (*cb_pari_sigint)(void)`. Function called when we receive `SIGINT`. By default, raises

```
pari_err(e_MISC, "user interrupt");
```

A possible simple-minded variant, used by the `gp` interpreter, is

`void gp_sigint_fun(void)`

`void (*cb_pari_pre_recover)(long)` initialized to `NULL`. If not `NULL`, this routine is called just before PARI cleans up from an error. It is not required to return. The error number is passed as argument.

`void (*cb_pari_err_recover)(long)` initialized to `pari_exit()`. This callback must not return. It is called after PARI has cleaned-up from an error. The error number is passed as argument, unless the PARI stack has been destroyed, in which case it is called with argument -1.

`int (*cb_pari_whatnow)(PariOUT *out, const char *s, int flag)` initialized to `NULL`. If not `NULL`, must check whether s existed in older versions of `pari` (the `gp` callback checks against `pari-1.39.15`). All output must be done via `out` methods.

- *flag* $= 0$: should print verbosely the answer, including help text if available.

- *flag* $= 1$: must return 0 if the function did not change, and a nonzero result otherwise. May print a help message.

5.1.5 Configuration variables.

`pari_library_path`: If set, It should be a path to the libpari library. It is used by the function `gpinstall` to locate the PARI library when searching for symbols. This should only be useful on Windows.

5.1.6 Utility functions.

`void pari_ask_confirm(const char *s)` raise an error if the callback `cb_pari_ask_confirm` is `NULL`. Otherwise calls

```
cb_pari_ask_confirm(s);
```

`char* gp_filter(const char *s)` pre-processor for the GP parser: filter out whitespace and GP comments from s. The returned string is allocated on the PARI stack and must not be freed.

`GEN pari_compile_str(const char *s)` low-level form of `compile_str`: assumes that s does not contain spaces or GP comments and returns the closure attached to the GP expression s. Note that GP metacommands are not recognized.

`int gp_meta(const char *s, int ismain)` low-level component of `gp_read_str`: assumes that s does not contain spaces or GP comments and try to interpret s as a GP metacommand (e.g. starting by \ or `?`). If successful, execute the metacommand and return 1; otherwise return 0. The `ismain` parameter modifies the way `\r` commands are handled: if nonzero, act as if the file contents were entered via standard input (i.e. call `switchin` and divert `pari_infile`); otherwise, simply call `gp_read_file`.

`void pari_hit_return(void)` wait for the use to enter `\n` via standard input.

`void gp_load_gprc(void)` read and execute the user's `GPRC` file.

`void pari_center(const char *s)` print s, centered.

`void pari_print_version(void)` print verbose version information.

`long pari_community(void)` return the index of the support section n the help.

`const char* gp_format_time(long t)` format a delay of t ms suitable for `gp` output, with `timer` set. The string is allocated in the PARI stack via `stack_malloc`.

`const char* gp_format_prompt(const char *p)` format a prompt p suitable for `gp` prompting (includes colors and protecting ANSI escape sequences for readline).

`void pari_alarm(long s)` set an alarm after s seconds (raise an `e_ALARM` exception).

`void gp_help(const char *s, long flag)` print help for s, depending on the value of *flag*:

- `h_REGULAR`, basic help (`?`);
- `h_LONG`, extended help (`??`);
- `h_APROPOS`, a propos help (`??`).

`const char ** gphelp_keyword_list(void)` return a `NULL`-terminated array a strings, containing keywords known to `gphelp` besides GP functions (e.g. `modulus` or `operator`). Used by the online help system and the contextual completion engine.

`void gp_echo_and_log(const char *p, const char *s)` given a prompt p and attached input command s, update logfile and possibly print on standard output if `echo` is set and we are not in interactive mode. The callback `cb_pari_is_interactive` must be set to a sensible value.

`void gp_alarm_handler(int sig)` the `SIGALRM` handler set by the `gp` interpreter.

`void print_fun_list(char **list, long n)` print all elements of `list` in columns, pausing (hit return) every n lines. `list` is `NULL` terminated.

5.1.7 Saving and restoring the GP context.

`void gp_context_save(struct gp_context* rec)` save the current GP context.

`void gp_context_restore(struct gp_context* rec)` restore a GP context. The new context must be an ancestor of the current context.

5.1.8 GP history.

These functions allow to control the GP history (the `%` operator).

`void pari_add_hist(GEN x, long t, long r)` adds `x` as the last history entry; t (resp. r) is the cpu (resp. real) time used to compute it.

`GEN pari_get_hist(long p)`, if $p > 0$ returns entry of index p (i.e. `%p`), else returns entry of index $n+p$ where n is the index of the last entry (used for `%`, `%‘`, `%‘‘`, etc.).

`long pari_get_histtime(long p)` as `pari_get_hist`, returning the cpu time used to compute the history entry, instead of the entry itself.

`long pari_get_histrtime(long p)` as `pari_get_hist`, returning the real time used to compute the history entry, instead of the entry itself.

`GEN pari_histtime(long p)` return the vector `[cpu, real]` where `cpu` and `real` are as above.

`ulong pari_nb_hist(void)` return the index of the last entry.

5.2 Handling GENs.

Almost all these functions are either macros or inlined. Unless mentioned otherwise, they do not evaluate their arguments twice. Most of them are specific to a set of types, although no consistency checks are made: e.g. one may access the `sign` of a `t_PADIC`, but the result is meaningless.

5.2.1 Allocation.

`GEN cgetg(long l, long t)` allocates (the root of) a `GEN` of type t and length l. Sets $z[0]$.

`GEN cgeti(long l)` allocates a `t_INT` of length l (including the 2 codewords). Sets $z[0]$ only.

`GEN cgetr(long l)` allocates a `t_REAL` of length l (including the 2 codewords). Sets $z[0]$ only.

`GEN cgetc(long prec)` allocates a `t_COMPLEX` whose real and imaginary parts are `t_REAL`s of length `prec`.

`GEN cgetg_copy(GEN x, long *lx)` fast version of `cgetg`: allocate a `GEN` with the same type and length as x, setting `*lx` to `lg(x)` as a side-effect. (Only sets the first codeword.) This is a little faster than `cgetg` since we may reuse the bitmask in $x[0]$ instead of recomputing it, and we do not need to check that the length does not overflow the possibilities of the implementation (since an object with that length already exists). Note that `cgetg` with arguments known at compile time, as in

```
cgetg(3, t_INTMOD)
```

will be even faster since the compiler will directly perform all computations and checks.

`GEN vectrunc_init(long l)` perform `cgetg(l,t_VEC)`, then set the length to 1 and return the result. This is used to implement vectors whose final length is easily bounded at creation time, that we intend to fill gradually using:

`void vectrunc_append(GEN x, GEN y)` assuming x was allocated using `vectrunc_init`, appends y as the last element of x, which grows in the process. The function is shallow: we append y, not a copy; it is equivalent to

```
long lx = lg(x); gel(x,lx) = y; setlg(x, lx+1);
```

Beware that the maximal size of x (the l argument to `vectrunc_init`) is unknown, hence unchecked, and stack corruption will occur if we append more than $l-1$ elements to x. Use the safer (but slower) `shallowconcat` when l is not easy to bound in advance.

An other possibility is simply to allocate using `cgetg(l, t)` then fill the components as they become available: this time the downside is that we do not obtain a correct `GEN` until the vector is complete. Almost no PARI function will be able to operate on it.

`void vectrunc_append_batch(GEN x, GEN y)` successively apply

```
vectrunc_append(x, gel(y, i))
```

for all elements of the vector y.

`GEN coltrunc_init(long l)` as `vectrunc_init` but perform `cgetg(l,t_COL)`.

`GEN vecsmalltrunc_init(long l)`

`void vecsmalltrunc_append(GEN x, long t)` analog to the above for a `t_VECSMALL` container.

5.2.2 Length conversions.

These routines convert a nonnegative length to different units. Their behavior is undefined at negative integers.

`long ndec2nlong(long x)` converts a number of decimal digits to a number of words. Returns $1 + \texttt{floor}(x \times \texttt{BITS\_IN\_LONG} \log_2 10)$.

`long ndec2prec(long x)` converts a number of decimal digits to a number of codewords. This is equal to 2 + `ndec2nlong(x)`.

`long ndec2nbits(long x)` convers a number of decimal digits to a number of bits.

`long prec2ndec(long x)` converts a number of codewords to a number of decimal digits.

`long nbits2nlong(long x)` converts a number of bits to a number of words. Returns the smallest word count containing x bits, i.e $\texttt{ceil}(x/\texttt{BITS\_IN\_LONG})$.

`long nbits2ndec(long x)` converts a number of bits to a number of decimal digits.

`long nbits2lg(long x)` converts a number of bits to a length in code words. Currently an alias for `nbits2nlong`.

`long nbits2prec(long x)` converts a number of bits to a number of codewords. This is equal to 2 + `nbits2nlong(x)`.

`long nbits2extraprec(long x)` converts a number of bits to the mantissa length of a `t_REAL` in codewords. This is currently an alias to `nbits2nlong(x)`.

`long nchar2nlong(long x)` converts a number of bytes to number of words. Returns the smallest word count containing x bytes, i.e $\texttt{ceil}(x/\texttt{sizeof}(\texttt{long}))$.

`long prec2nbits(long x)` converts a `t_REAL` length into a number of significant bits; returns $(x - 2)\texttt{BITS\_IN\_LONG}$.

`double prec2nbits_mul(long x, double y)` returns $\texttt{prec2nbits}(x) \times y$.

`long bit_accuracy(long x)` converts a length into a number of significant bits; currently an alias for `prec2nbits`.

`double bit_accuracy_mul(long x, double y)` returns $\texttt{bit\_accuracy}(x) \times y$.

`long realprec(GEN x)` length of a `t_REAL` in words; currently an alias for `lg`.

`long bit_prec(GEN x)` length of a `t_REAL` in bits.

`long precdbl(long prec)` given a length in words corresponding to a `t_REAL` precision, return the length corresponding to doubling the precision. Due to the presence of 2 code words, this is $2(\texttt{prec} - 2) + 2$.

5.2.3 Read type-dependent information.

`long typ(GEN x)` returns the type number of `x`. The header files included through `pari.h` define symbolic constants for the `GEN` types: `t_INT` etc. Never use their actual numerical values. E.g to determine whether `x` is a `t_INT`, simply check

```
if (typ(x) == t_INT) { }
```

The types are internally ordered and this simplifies the implementation of commutative binary operations (e.g addition, gcd). Avoid using the ordering directly, as it may change in the future; use type grouping functions instead (Section 5.2.6).

`const char* type_name(long t)` given a type number `t` this routine returns a string containing its symbolic name. E.g `type_name(t_INT)` returns `"t_INT"`. The return value is read-only.

`long lg(GEN x)` returns the length of `x` in `BITS_IN_LONG`-bit words.

`long lgefint(GEN x)` returns the effective length of the `t_INT` `x` in `BITS_IN_LONG`-bit words.

`long signe(GEN x)` returns the sign (-1, 0 or 1) of `x`. Can be used for `t_INT`, `t_REAL`, `t_POL` and `t_SER` (for the last two types, only 0 or 1 are possible).

`long gsigne(GEN x)` returns the sign of a real number x, valid for `t_INT`, `t_REAL` as `signe`, but also for `t_FRAC` and `t_QUAD` of positive discriminants. Raise a type error if `typ(x)` is not among those.

`long expi(GEN x)` returns the binary exponent of the real number equal to the `t_INT` `x`. This is a special case of `gexpo`.

`long expo(GEN x)` returns the binary exponent of the `t_REAL` `x`.

`long mpexpo(GEN x)` returns the binary exponent of the `t_INT` or `t_REAL` `x`.

`long gexpo(GEN x)` same as `expo`, but also valid when `x` is not a `t_REAL` (returns the largest exponent found among the components of `x`). When `x` is an exact 0, this returns `-HIGHEXPOBIT`, which is lower than any valid exponent.

`long gexpo_safe(GEN x)` same as `gexpo`, but returns a value strictly less than `-HIGHEXPOBIT` when the exponent is not defined (e.g. for a `t_PADIC` or `t_INTMOD` component).

`long valp(GEN x)` returns the p-adic valuation (for a `t_PADIC`) or X-adic valuation (for a `t_SER`, taken with respect to the main variable) of `x`.

`long precp(GEN x)` returns the precision of the `t_PADIC` `x`.

`long varn(GEN x)` returns the variable number of the `t_POL` or `t_SER` `x` (between 0 and `MAXVARN`).

`long gvar(GEN x)` returns the main variable number when any variable at all occurs in the composite object `x` (the smallest variable number which occurs), and `NO_VARIABLE` otherwise.

`long gvar2(GEN x)` returns the variable number for the ring over which x is defined, e.g. if $x \in \mathbf{Z}[a][b]$ return (the variable number for) a. Return `NO_VARIABLE` if x has no variable or is not defined over a polynomial ring.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL` `x` with respect to its main variable, *if* its leading coefficient is nonzero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. `Mod(0,2)`). If x has no coefficients (rational 0 polynomial), its length is 2 and we return the expected -1.

`long lgpol(GEN x)` is equal to `degpol(x) + 1`. Used to loop over the coefficients of a `t_POL` in the following situation:

```
GEN xd = x + 2;
long i, l = lgpol(x);
for (i = 0; i < l; i++) foo( xd[i] ).
```

`long precision(GEN x)` If `x` is of type `t_REAL`, returns the precision of `x`, namely

• if `x` is not zero: the length of `x` in `BITS_IN_LONG`-bit words;

• if `x` is numerically equal to 0, of exponent e: the absolute accuracy `nbits2prec`(e) if $e < 0$ and `LOWDEFAULTPREC` if $e \geq 0$.

If `x` is of type `t_COMPLEX`, returns the minimum of the precisions of the real and imaginary part. Otherwise, returns 0 (which stands for infinite precision). In all cases, the precision is either 0 or can be used as a `prec` parameter in transcendental functions.

`long lgcols(GEN x)` is equal to `lg(gel(x,1))`. This is the length of the columns of a `t_MAT` with at least one column.

`long nbrows(GEN x)` is equal to `lg(gel(x,1))-1`. This is the number of rows of a `t_MAT` with at least one column.

`long gprecision(GEN x)` as `precision` for scalars. Returns the lowest precision encountered among the components otherwise.

`long sizedigit(GEN x)` returns 0 if `x` is exactly 0. Otherwise, returns **gexpo**`(x)` multiplied by $\log_{10}(2)$. This gives a crude estimate for the maximal number of decimal digits of the components of `x`.

5.2.4 Eval type-dependent information. These routines convert type-dependent information to bitmask to fill the codewords of `GEN` objects (see Section 4.5). E.g for a `t_REAL` `z`:

```
z[1] = evalsigne(-1) | evalexpo(2)
```

Compatible components of a codeword for a given type can be OR-ed as above.

`ulong evaltyp(long x)` convert type `x` to bitmask (first codeword of all `GEN`s)

`long evallg(long x)` convert length `x` to bitmask (first codeword of all `GEN`s). Raise overflow error if `x` is so large that the corresponding length cannot be represented

`long _evallg(long x)` as `evallg` *without* the overflow check.

`ulong evalvarn(long x)` convert variable number `x` to bitmask (second codeword of `t_POL` and `t_SER`)

`long evalsigne(long x)` convert sign `x` (in $-1, 0, 1$) to bitmask (second codeword of `t_INT`, `t_REAL`, `t_POL`, `t_SER`)

`long evalprecp(long x)` convert p-adic (X-adic) precision `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding precision cannot be represented.

`long _evalprecp(long x)` same as `evalprecp` *without* the overflow check.

`long evalvalp(long x)` convert p-adic (X-adic) valuation `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding valuation cannot be represented.

`long _evalvalp(long x)` same as `evalvalp` *without* the overflow check.

`long evalexpo(long x)` convert exponent `x` to bitmask (second codeword of `t_REAL`). Raise overflow error if `x` is so large that the corresponding exponent cannot be represented

`long _evalexpo(long x)` same as `evalexpo` *without* the overflow check.

`long evallgefint(long x)` convert effective length `x` to bitmask (second codeword `t_INT`). This should be less or equal than the length of the `t_INT`, hence there is no overflow check for the effective length.

5.2.5 Set type-dependent information. Use these functions and macros with extreme care since usually the corresponding information is set otherwise, and the components and further codeword fields (which are left unchanged) may not be compatible with the new information.

`void settyp(GEN x, long s)` sets the type number of `x` to `s`.

`void setlg(GEN x, long s)` sets the length of `x` to `s`. This is an efficient way of truncating vectors, matrices or polynomials.

`void setlgefint(GEN x, long s)` sets the effective length of the `t_INT` `x` to `s`. The number `s` must be less than or equal to the length of `x`.

`void setsigne(GEN x, long s)` sets the sign of `x` to `s`. If `x` is a `t_INT` or `t_REAL`, `s` must be equal to -1, 0 or 1, and if `x` is a `t_POL` or `t_SER`, `s` must be equal to 0 or 1. No sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void togglesign(GEN x)` sets the sign s of `x` to $-s$, in place.

`void togglesign_safe(GEN *x)` sets the s sign of `*x` to $-s$, in place, unless `*x` is one of the integer universal constants in which case replace `*x` by its negation (e.g. replace `gen_1` by `gen_m1`).

`void setabssign(GEN x)` sets the sign s of `x` to $|s|$, in place.

`void affectsign(GEN x, GEN y)` shortcut for `setsigne(y, signe(x))`. No sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void affectsign_safe(GEN x, GEN *y)` sets the sign of `*y` to that of `x`, in place, unless `*y` is one of the integer universal constants in which case replace `*y` by its negation if needed (e.g. replace `gen_1` by `gen_m1` if `x` is negative). No other sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void normalize_frac(GEN z)` assuming z is of the form `mkfrac(a,b)` with $b \neq 0$, make sure that $b > 0$ by changing the sign of a in place if needed (use `togglesign`).

`void setexpo(GEN x, long s)` sets the binary exponent of the `t_REAL` `x` to `s`. The value `s` must be a 24-bit signed number.

`void setvalp(GEN x, long s)` sets the p-adic or X-adic valuation of `x` to `s`, if `x` is a `t_PADIC` or a `t_SER`, respectively.

`void setprecp(GEN x, long s)` sets the p-adic precision of the `t_PADIC` `x` to `s`.

`void setvarn(GEN x, long s)` sets the variable number of the `t_POL` or `t_SER` `x` to `s` (where $0 \leq$ `s` $\leq$ `MAXVARN`).

5.2.6 Type groups. In the following functions, `t` denotes the type of a `GEN`. They used to be implemented as macros, which could evaluate their argument twice; *no longer*: it is not inefficient to write

```
is_intreal_t(typ(x))
```

`int is_recursive_t(long t)` true iff `t` is a recursive type (the nonrecursive types are `t_INT`, `t_REAL`, `t_STR`, `t_VECSMALL`). Somewhat contrary to intuition, `t_LIST` is also nonrecursive, ; see the Developer's guide for details.

`int is_intreal_t(long t)` true iff `t` is `t_INT` or `t_REAL`.

`int is_rational_t(long t)` true iff `t` is `t_INT` or `t_FRAC`.

`int is_real_t(long t)` true iff `t` is `t_INT` or `t_REAL` or `t_FRAC`.

`int is_qfb_t(long t)` true iff `t` is `t_QFB`.

`int is_vec_t(long t)` true iff `t` is `t_VEC` or `t_COL`.

`int is_matvec_t(long t)` true iff `t` is `t_MAT`, `t_VEC` or `t_COL`.

`int is_scalar_t(long t)` true iff `t` is a scalar, i.e a `t_INT`, a `t_REAL`, a `t_INTMOD`, a `t_FRAC`, a `t_COMPLEX`, a `t_PADIC`, a `t_QUAD`, or a `t_POLMOD`.

`int is_extscalar_t(long t)` true iff `t` is a scalar (see `is_scalar_t`) or `t` is `t_POL`.

`int is_const_t(long t)` true iff `t` is a scalar which is not `t_POLMOD`.

`int is_noncalc_t(long t)` true if generic operations (`gadd`, `gmul`) do not make sense for t: corresponds to types `t_LIST`, `t_STR`, `t_VECSMALL`, `t_CLOSURE`

5.2.7 Accessors and components. The first two functions return `GEN` components as copies on the stack:

`GEN compo(GEN x, long n)` creates a copy of the `n`-th true component (i.e. not counting the codewords) of the object `x`.

`GEN truecoeff(GEN x, long n)` creates a copy of the coefficient of degree `n` of `x` if `x` is a scalar, `t_POL` or `t_SER`, and otherwise of the `n`-th component of `x`.

On the contrary, the following routines return the address of a `GEN` component. No copy is made on the stack:

`GEN constant_coeff(GEN x)` returns the address of the constant coefficient of `t_POL x`. By convention, a 0 polynomial (whose `sign` is 0) has `gen_0` constant term.

`GEN leading_coeff(GEN x)` returns the address of the leading coefficient of `t_POL x`, i.e. the coefficient of largest index stored in the array representing x. This may be an inexact 0. By convention, return `gen_0` if the coefficient array is empty.

`GEN gel(GEN x, long i)` returns the address of the `x[i]` entry of `x`. (`el` stands for element.)

`GEN gcoeff(GEN x, long i, long j)` returns the address of the `x[i,j]` entry of `t_MAT x`, i.e. the coefficient at row `i` and column `j`.

`GEN gmael(GEN x, long i, long j)` returns the address of the `x[i][j]` entry of `x`. (`mael` stands for multidimensional array element.)

`GEN gmael2(GEN A, long x1, long x2)` is an alias for `gmael`. Similar macros `gmael3`, `gmael4`, `gmael5` are available.

5.3 Global numerical constants.

These are defined in the various public PARI headers.

5.3.1 Constants related to word size.

long BITS_IN_LONG $= 2^{\texttt{TWOPOTBITS\_IN\_LONG}}$: number of bits in a long (32 or 64).

long BITS_IN_HALFULONG: BITS_IN_LONG divided by 2.

long LONG_MAX: the largest positive long.

ulong ULONG_MAX: the largest ulong.

long DEFAULTPREC: the length (lg) of a t_REAL with 64 bits of accuracy

long MEDDEFAULTPREC: the length (lg) of a t_REAL with 128 bits of accuracy

long BIGDEFAULTPREC: the length (lg) of a t_REAL with 192 bits of accuracy

ulong HIGHBIT: the largest power of 2 fitting in an ulong.

ulong LOWMASK: bitmask yielding the least significant bits.

ulong HIGHMASK: bitmask yielding the most significant bits.

The last two are used to implement the following convenience macros, returning half the bits of their operand:

ulong LOWWORD(ulong a) returns least significant bits.

ulong HIGHWORD(ulong a) returns most significant bits.

Finally

long divsBIL(long n) returns the Euclidean quotient of n by BITS_IN_LONG (with nonnegative remainder).

long remsBIL(n) returns the (nonnegative) Euclidean remainder of n by BITS_IN_LONG

long dvmdsBIL(long n, long *r)

ulong dvmduBIL(ulong n, ulong *r) sets r to remsBIL(n) and returns divsBIL(n).

5.3.2 Masks used to implement the GEN type.

These constants are used by higher level macros, like typ or lg:

EXPOnumBITS, LGnumBITS, SIGNnumBITS, TYPnumBITS, VALPnumBITS, VARNnumBITS: number of bits used to encode expo, lg, signe, typ, valp, varn.

PRECPSHIFT, SIGNSHIFT, TYPSHIFT, VARNSHIFT: shifts used to recover or encode precp, varn, typ, signe

CLONEBIT, EXPOBITS, LGBITS, PRECPBITS, SIGNBITS, TYPBITS, VALPBITS, VARNBITS: bitmasks used to extract isclone, expo, lg, precp, signe, typ, valp, varn from GEN codewords.

MAXVARN: the largest possible variable number.

NO_VARIABLE: sentinel returned by gvar(x) when x does not contain any polynomial; has a lower priority than any valid variable number.

HIGHEXPOBIT: a power of 2, one more that the largest possible exponent for a t_REAL.

HIGHVALPBIT: a power of 2, one more that the largest possible valuation for a t_PADIC or a t_SER.

5.3.3 $\log 2$, π.

These are `double` approximations to useful constants:

`M_PI`: π.

`M_LN2`: $\log 2$.

`LOG10_2`: $\log 2/\log 10$.

`LOG2_10`: $\log 10/\log 2$.

5.4 Iterating over small primes, low-level interface.

One of the methods used by the high-level prime iterator (see Section 4.8.2), is a precomputed table. Its direct use is deprecated, but documented here.

After `pari_init(size, maxprime)`, a "prime table" is initialized with the successive *differences* of primes up to (possibly just a little beyond) `maxprime`. The prime table occupies roughly `maxprime`/ log(`maxprime`) bytes in memory, so be sensible when choosing `maxprime`; it is 500000 by default under `gp` and there is no real benefit in choosing a much larger value: the high-level iterator provide *fast* access to primes up to the *square* of `maxprime`. In any case, the implementation requires that `maxprime` $< 2^{\texttt{BITS\_IN\_LONG}} - 2048$, whatever memory is available.

PARI currently guarantees that the first 6547 primes, up to and including 65557, are present in the table, even if you set `maxprime` to zero. in the `pari_init` call.

Some convenience functions:

`ulong maxprime()` the largest prime computable using our prime table.

`ulong maxprimeN()` the index N of the largest prime computable using the prime table. I.e., $p_N =$ `maxprime`().

`void maxprime_check(ulong B)` raise an error if `maxprime()` is $< B$.

After the following initializations (the names p and ptr are arbitrary of course)

```
byteptr ptr = diffptr;
ulong p = 0;
```

calling the macro `NEXT_PRIME_VIADIFF_CHECK`(p, ptr) repeatedly will assign the successive prime numbers to p. Overrunning the prime table boundary will raise the error `e_MAXPRIME`, which just prints the error message:

```
*** not enough precomputed primes, need primelimit ~c
```

(for some numerical value c), then the macro aborts the computation. The alternative macro `NEXT_PRIME_VIADIFF` operates in the same way, but will omit that check, and is slightly faster. It should be used in the following way:

```
byteptr ptr = diffptr;
ulong p = 0;

if (maxprime() < goal) pari_err_MAXPRIME(goal); /* not enough primes */
while (p <= goal) /* run through all primes up to goal */
{
  NEXT_PRIME_VIADIFF(p, ptr);
```

```
  ...
}
```

Here, we use the general error handling function `pari_err` (see Section 4.7.3), with the codeword `e_MAXPRIME`, raising the "not enough primes" error. This could be rewritten as

```
maxprime_check(goal);
while (p <= goal) /* run through all primes up to goal */
{
  NEXT_PRIME_VIADIFF(p, ptr);
  ...
}
```

`bytepr initprimes(ulong maxprime, long *L, ulong *lastp)` computes a (malloc'ed) "prime table", in fact a table of all prime differences for $p <$ `maxprime` (and possibly a little beyond). Set L to the table length (argument to `malloc`), and *lastp* to the last prime in the table.

`void initprimetable(ulong maxprime)` computes a prime table (of all prime differences for $p <$ `maxprime`) and assign it to the global variable `diffptr`. Don't change `diffptr` directly, call this function instead. This calls `initprimes` and updates internal data recording the table size.

`ulong init_primepointer_geq(ulong a, byteptr *pd)` returns the smallest prime $p \geq a$, and sets `*pd` to the proper offset of `diffptr` so that `NEXT_PRIME_VIADIFF(p, *pd)` correctly returns `unextprime(p + 1)`.

`ulong init_primepointer_gt(ulong a, byteptr *pd)` returns the smallest prime $p > a$.

`ulong init_primepointer_leq(ulong a, byteptr *pd)` returns the largest prime $p \leq a$.

`ulong init_primepointer_lt(ulong a, byteptr *pd)` returns the largest prime $p < a$.

5.5 Handling the PARI stack.

5.5.1 Allocating memory on the stack.

`GEN cgetg(long n, long t)` allocates memory on the stack for an object of length `n` and type `t`, and initializes its first codeword.

`GEN cgeti(long n)` allocates memory on the stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_INT)`.

`GEN cgetr(long n)` allocates memory on the stack for a `t_REAL` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_REAL)`.

`GEN cgetc(long n)` allocates memory on the stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REAL`s of length `n`.

`GEN cgetp(GEN x)` creates space sufficient to hold the `t_PADIC` `x`, and sets the prime p and the p-adic precision to those of `x`, but does not copy (the p-adic unit or zero representative and the modulus of) `x`.

`GEN new_chunk(size_t n)` allocates a `GEN` with n components, *without* filling the required code words. This is the low-level constructor underlying `cgetg`, which calls `new_chunk` then sets the first code word. It works by simply returning the address `((GEN)avma) - n`, after checking that it is larger than `(GEN)bot`.

`void new_chunk_resize(size_t x)` this function is called by `new_chunk` when the PARI stack overflows. There is no need to call it manually. It will either extend the stack or report an `e_STACK` error.

`char* stack_malloc(size_t n)` allocates memory on the stack for n chars (*not* n `GEN`s). This is faster than using `malloc`, and easier to use in most situations when temporary storage is needed. In particular there is no need to `free` individually all variables thus allocated: a simple `set_avma(oldavma)` might be enough. On the other hand, beware that this is not permanent independent storage, but part of the stack. The memory is aligned on `sizeof(long)` bytes boundaries.

`char* stack_malloc_align(size_t n, long k)` as `stack_malloc`, but the memory is aligned on `k` bytes boundaries. The numberk must be a multiple of the `sizeof(long)`.

`char* stack_calloc(size_t n)` as `stack_malloc`, setting the memory to zero.

`char* stack_calloc_align(size_t n, long k)` as `stack_malloc_align`, setting the memory to zero.

Objects allocated through these last three functions cannot be `gerepile`'d, since they are not yet valid `GEN`s: their codewords must be filled first.

`GEN cgetalloc(long t, size_t l)`, same as `cgetg(t, l)`, except that the result is allocated using `pari_malloc` instead of the PARI stack. The resulting `GEN` is now impervious to garbage collecting routines, but should be freed using `pari_free`.

5.5.2 Stack-independent binary objects.

`GENbin* copy_bin(GEN x)` copies x into a malloc'ed structure suitable for stack-independent binary transmission or storage. The object obtained is architecture independent provided, `sizeof(long)` remains the same on all PARI instances involved, as well as the multiprecision kernel (either native or GMP).

`GENbin* copy_bin_canon(GEN x)` as `copy_bin`, ensuring furthermore that the binary object is independent of the multiprecision kernel. Slower than `copy_bin`.

`GEN bin_copy(GENbin *p)` assuming p was created by `copy_bin(x)` (not necessarily by the same PARI instance: transmission or external storage may be involved), restores x on the PARI stack.

The routine `bin_copy` transparently encapsulate the following functions:

`GEN GENbinbase(GENbin *p)` the `GEN` data actually stored in p. All addresses are stored as offsets with respect to a common reference point, so the resulting `GEN` is unusable unless it is a nonrecursive type; private low-level routines must be called first to restore absolute addresses.

`void shiftaddress(GEN x, long dec)` converts relative addresses to absolute ones.

`void shiftaddress_canon(GEN x, long dec)` converts relative addresses to absolute ones, and converts leaves from a canonical form to the one specific to the multiprecision kernel in use. The `GENbin` type stores whether leaves are stored in canonical form, so `bin_copy` can call the right variant.

Objects containing closures are harder to e.g. copy and save to disk, since closures contain pointers to libpari functions that will not be valid in another gp instance: there is little chance for them to be loaded at the exact same address in memory. Such objects must be saved along with a linking table.

`GEN copybin_unlink(GEN C)` returns a linking table allowing to safely store and transmit `t_CLOSURE` objects in C. If C = `NULL` return a linking table corresponding to the content of all gp variables. C may then be dumped to disk in binary form, for instance.

`void bincopy_relink(GEN C, GEN V)` given a binary object C, as dumped by writebin and read back into a session, and a linking table V, restore all closures contained in C (function pointers are translated to their current value).

5.5.3 Garbage collection. See Section 4.3 for a detailed explanation and many examples.

`void set_avma(ulong av)` reset `avma` to `av`. You may think of this as a simple `avma = av` statement, but PARI developpers modify this statement in special code branches to detect garbage collecting issues (by invalidating the PARI stack below `av`).

`ulong get_avma(void)` return `avma`. Useful for languages that do not provide access to TLS variables.

`GEN gc_NULL(pari_sp av)` reset `avma` to `av` and return `NULL`.

The following 6 functions reset `avma` to `av` and return x:

`int gc_bool(pari_sp av, int x)`

`double gc_double(pari_sp av, double x)`

`int gc_int(pari_sp av, int x)`

`long gc_long(pari_sp av, long x)`

`ulong gc_ulong(pari_sp av, ulong x)` This allows for instance to return `gc_ulong(av, itou(z))`, whereas

```
pari_sp av = avma;
GEN z = ...
set_avma(av);
return itou(z);
```

should be frowned upon since `set_avma(av)` conceptually destroys everything from the reference point on, including z.

`GEN gc_const(pari_sp av, GEN x)` assumes that x is either not on the stack (clone, universal constant such as `gen_0`) or was defined before `av`.

`GEN gc_all(pari_sp av, int n, ...)`. Assumes that $1 \leq n \leq 10$; This is similar to `gerepileall`, expecting n further `GEN*` arguments: the stack is cleaned and the corresponding `GEN` are copied to the stack starting from `av` (in this order: the first argument comes first), and the first such `GEN` is returned. To be used in the following scenario:

```
GEN f(..., GEN *py)
{
  pari_sp av = avma;
  GEN x = ..., y = ...
  *py = y; return gc_all(av, 2, &x, py);
}
```

This function returns x, and the user also recovers y as a side effect. Not that we can later use `cgiv`(y) to recover the memory used by y while still keeping x.

`void cgiv(GEN x)` frees object `x`, assuming it is the last created on the stack.

`GEN gerepile(pari_sp p, pari_sp q, GEN x)` general garbage collector for the stack.

`void gerepileall(pari_sp av, int n, ...)` cleans up the stack from `av` on (i.e from `avma` to `av`), preserving the `n` objects which follow in the argument list (of type `GEN*`). For instance, `gerepileall(av, 2, &x, &y)` preserves `x` and `y`.

`void gerepileallsp(pari_sp av, pari_sp ltop, int n, ...)` cleans up the stack between `av` and `ltop`, updating the `n` elements which follow `n` in the argument list (of type `GEN*`). Check that the elements of `g` have no component between `av` and `ltop`, and assumes that no garbage is present between `avma` and `ltop`. Analogous to (but faster than) `gerepileall` otherwise.

`GEN gerepilecopy(pari_sp av, GEN x)` cleans up the stack from `av` on, preserving the object `x`. Special case of `gerepileall` (case `n` = 1), except that the routine returns the preserved `GEN` instead of updating its address through a pointer.

`void gerepilemany(pari_sp av, GEN* g[], int n)` alternative interface to `gerepileall`. The preserved `GEN`s are the elements of the array `g` of length n: `g[0]`, `g[1]`, …, `g[`n`-1]`. Obsolete: no more efficient than `gerepileall`, error-prone, and clumsy (need to declare an extra `GEN *g`).

`void gerepilemanysp(pari_sp av, pari_sp ltop, GEN* g[], int n)` alternative interface to `gerepileallsp`. Obsolete.

`void gerepilecoeffs(pari_sp av, GEN x, int n)` cleans up the stack from `av` on, preserving `x[0]`, …, `x[n-1]` (which are `GEN`s).

`void gerepilecoeffssp(pari_sp av, pari_sp ltop, GEN x, int n)` cleans up the stack from `av` to `ltop`, preserving `x[0]`, …, `x[n-1]` (which are `GEN`s). Same assumptions as in `gerepilemanysp`, of which this is a variant. For instance

```
z = cgetg(3, t_COMPLEX);
av = avma; garbage(); ltop = avma;
z[1] = fun1();
z[2] = fun2();
gerepilecoeffssp(av, ltop, z + 1, 2);
return z;
```

cleans up the garbage between `av` and `ltop`, and connects `z` and its two components. This is marginally more efficient than the standard

```
av = avma; garbage(); ltop = avma;
z = cgetg(3, t_COMPLEX);
z[1] = fun1();
z[2] = fun2(); return gerepile(av, ltop, z);
```

`GEN gerepileupto(pari_sp av, GEN q)` analogous to (but faster than) `gerepilecopy`. Assumes that `q` is connected and that its root was created before any component. If `q` is not on the stack, this is equivalent to `set_avma(av)`; in particular, sentinels which are not even proper `GEN`s such as `q = NULL` are allowed.

`GEN gerepileuptoint(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a `t_INT`. The length and effective length of the resulting `t_INT` are equal.

`GEN gerepileuptoleaf(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a leaf, i.e a nonrecursive type (`is_recursive_t(typ(q))` is nonzero).

Contrary to gerepileuptoint and gerepileupto, gerepileuptoleaf leaves length and effective length of a t_INT unchanged.

5.5.4 Garbage collection: advanced use.

void stackdummy(pari_sp av, pari_sp ltop) inhibits the memory area between av *included* and ltop *excluded* with respect to gerepile, in order to avoid a call to gerepile(av, ltop,...). The stack space is not reclaimed though.

More precisely, this routine assumes that av is recorded earlier than ltop, then marks the specified stack segment as a nonrecursive type of the correct length. Thus gerepile will not inspect the zone, at most copy it. To be used in the following situation:

```
av0 = avma; z = cgetg(t_VEC, 3);
gel(z,1) = HUGE(); av = avma; garbage(); ltop = avma;
gel(z,2) = HUGE(); stackdummy(av, ltop);
```

Compared to the orthodox

```
gel(z,2) = gerepile(av, ltop, gel(z,2));
```

or even more wasteful

```
z = gerepilecopy(av0, z);
```

we temporarily lose (av − ltop) words but save a costly gerepile. In principle, a garbage collection higher up the call chain should reclaim this later anyway.

Without the stackdummy, if the [av, ltop] zone is arbitrary (not even valid GENs as could happen after direct truncation via setlg), we would leave dangerous data in the middle of z, which would be a problem for a later

```
gerepile(..., ... , z);
```

And even if it were made of valid GENs, inhibiting the area makes sure gerepile will not inspect their components, saving time.

Another natural use in low-level routines is to "shorten" an existing GEN z to its first n − 1 components:

```
setlg(z, n);
stackdummy((pari_sp)(z + lg(z)), (pari_sp)(z + n));
```

or to its last n components:

```
long L = lg(z) - n, tz = typ(z);
stackdummy((pari_sp)(z + L), (pari_sp)z);
z += L; z[0] = evaltyp(tz) | evallg(L);
```

The first scenario (safe shortening an existing GEN) is in fact so common, that we provide a function for this:

void fixlg(GEN z, long ly) a safe variant of setlg(z, ly). If ly is larger than lg(z) do nothing. Otherwise, shorten z in place, using stackdummy to avoid later gerepile problems.

GEN gcopy_avma(GEN x, pari_sp *AVMA) return a copy of x as from gcopy, except that we pretend that initially avma is *AVMA, and that *AVMA is updated accordingly (so that the total size of x is the difference between the two successive values of `*AVMA`). It is not necessary for `*AVMA` to initially point on the stack: `gclone` is implemented using this mechanism.

`GEN icopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume x is a `t_INT` and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

5.5.5 Debugging the PARI stack.

`int chk_gerepileupto(GEN x)` returns 1 if `x` is suitable for `gerepileupto`, and 0 otherwise. In the latter case, print a warning explaining the problem.

`void dbg_gerepile(pari_sp ltop)` outputs the list of all objects on the stack between `avma` and `ltop`, i.e. the ones that would be inspected in a call to `gerepile(...,ltop,...)`.

`void dbg_gerepileupto(GEN q)` outputs the list of all objects on the stack that would be inspected in a call to `gerepileupto(...,q)`.

5.5.6 Copies.

`GEN gcopy(GEN x)` creates a new copy of x on the stack.

`GEN gcopy_lg(GEN x, long l)` creates a new copy of x on the stack, pretending that `lg(x)` is l, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `gcopy(x)`.

`int isonstack(GEN x)` `true` iff x belongs to the stack.

`void copyifstack(GEN x, GEN y)` sets `y = gcopy(x)` if x belongs to the stack, and `y = x` otherwise. This macro evaluates its arguments once, contrary to

```
y = isonstack(x)? gcopy(x): x;
```

`void icopyifstack(GEN x, GEN y)` as `copyifstack` assuming `x` is a `t_INT`.

5.5.7 Simplify.

`GEN simplify(GEN x)` you should not need that function in library mode. One rather uses:

`GEN simplify_shallow(GEN x)` shallow, faster, version of `simplify`.

5.6 The PARI heap.

5.6.1 Introduction.

It is implemented as a doubly-linked list of `malloc`'ed blocks of memory, equipped with reference counts. Each block has type `GEN` but need not be a valid `GEN`: it is a chunk of data preceded by a hidden header (meaning that we allocate x and return $x +$ `headersize`). A *clone*, created by `gclone`, is a block which is a valid `GEN` and whose *clone bit* is set.

5.6.2 Public interface.

`GEN newblock(size_t n)` allocates a block of n *words* (not bytes).

`void killblock(GEN x)` deletes the block x created by `newblock`. Fatal error if x not a block.

`GEN gclone(GEN x)` creates a new permanent copy of x on the heap (allocated using `newblock`). The *clone bit* of the result is set.

`GEN gcloneref(GEN x)` if x is not a clone, clone it and return the result; otherwise, increase the clone reference count and return x.

`void gunclone(GEN x)` deletes a clone. Deletion at first only decreases the reference count by 1. If the count remains positive, no further action is taken; if the count becomes zero, then the clone is actually deleted. In the current implementation, this is an alias for `killblock`, but it is cleaner to kill clones (valid `GEN`s) using this function, and other blocks using `killblock`.

`void guncloneNULL(GEN x)` same as `gunclone`, first checking whether x is `NULL` (and doing nothing in this case).

`void gunclone_deep(GEN x)` is only useful in the context of the GP interpreter which may replace arbitrary components of container types (`t_VEC`, `t_COL`, `t_MAT`, `t_LIST`) by clones. If x is such a container, the function recursively deletes all clones among the components of x, then unclones x. Useless in library mode: simply use `gunclone`.

`void guncloneNULL_deep(GEN x)` same as `gunclone_deep`, first checking whether x is `NULL` (and doing nothing in this case).

`void traverseheap(void(*f)(GEN, void *), void *data)` this applies `f(`x`, data)` to each object x on the PARI heap, most recent first. Mostly for debugging purposes.

`GEN getheap()` a simple wrapper around `traverseheap`. Returns a two-component row vector giving the number of objects on the heap and the amount of memory they occupy in long words.

`GEN cgetg_block(long x, long y)` as `cgetg(x,y)`, creating the return value as a `block`, not on the PARI stack.

`GEN cgetr_block(long prec)` as `cgetr(prec)`, creating the return value as a `block`, not on the PARI stack.

5.6.3 Implementation note. The hidden block header is manipulated using the following private functions:

`void* bl_base(GEN x)` returns the pointer that was actually allocated by `malloc` (can be freed).

`long bl_refc(GEN x)` the reference count of x: the number of pointers to this block. Decremented in `killblock`, incremented by the private function `void gclone_refc(GEN x)`; block is freed when the reference count reaches 0.

`long bl_num(GEN x)` the index of this block in the list of all blocks allocated so far (including freed blocks). Uniquely identifies a block until $2^{\texttt{BITS\_IN\_LONG}}$ blocks have been allocated and this wraps around.

`GEN bl_next(GEN x)` the block *after* x in the linked list of blocks (`NULL` if x is the last block allocated not yet killed).

`GEN bl_prev(GEN x)` the block allocated *before* x (never `NULL`).

We documented the last four routines as functions for clarity (and type checking) but they are actually macros yielding valid lvalues. It is allowed to write `bl_refc(x)++` for instance.

5.7 Handling user and temp variables.

Low-level implementation of user / temporary variables is liable to change. We describe it nevertheless for completeness. Currently variables are implemented by a single array of values divided in 3 zones: 0–`nvar` (user variables), `max_avail`–`MAXVARN` (temporary variables), and `nvar+1`–`max_avail-1` (pool of free variable numbers).

5.7.1 Low-level.

`void pari_var_init()`: a small part of `pari_init`. Resets variable counters `nvar` and `max_avail`, notwithstanding existing variables! In effect, this even deletes `x`. Don't use it.

`void pari_var_close(void)` attached destructor, called by `pari_close`.

`long pari_var_next()`: returns `nvar`, the number of the next user variable we can create.

`long pari_var_next_temp()` returns `max_avail`, the number of the next temp variable we can create.

`long pari_var_create(entree *ep)` low-level initialization of an `EpVAR`. Return the attached (new) variable number.

`GEN vars_sort_inplace(GEN z)` given a `t_VECSMALL` z of variable numbers, sort z in place according to variable priorities (highest priority comes first).

`GEN vars_to_RgXV(GEN h)` given a `t_VECSMALL` z of variable numbers, return the `t_VEC` of `pol_x`$(z[i])$.

5.7.2 User variables.

`long fetch_user_var(char *s)` returns a user variable whose name is `s`, creating it is needed (and using an existing variable otherwise). Returns its variable number.

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered v. Return `NULL` for a temporary variable.

`entree* is_entry(const char *s)` returns the `entree*` attached to an identifier `s` (variable or function), from the interpreter hashtables. Return `NULL` is the identifier is unknown.

5.7.3 Temporary variables.

`long fetch_var(void)` returns the number of a new temporary variable (decreasing `max_avail`).

`long delete_var(void)` delete latest temp variable created and return the number of previous one.

`void name_var(long n, char *s)` rename temporary variable number `n` to `s`; mostly useful for nicer printout. Error when trying to rename a user variable.

5.8 Adding functions to PARI.

5.8.1 Nota Bene. As mentioned in the COPYING file, modified versions of the PARI package can be distributed under the conditions of the GNU General Public License. If you do modify PARI, however, it is certainly for a good reason, and we would like to know about it, so that everyone can benefit from your changes. There is then a good chance that your improvements are incorporated into the next release.

We classify changes to PARI into four rough classes, where changes of the first three types are almost certain to be accepted. The first type includes all improvements to the documentation, in a broad sense. This includes correcting typos or inaccuracies of course, but also items which are not really covered in this document, e.g. if you happen to write a tutorial, or pieces of code exemplifying fine points unduly omitted in the present manual.

The second type is to expand or modify the configuration routines and skeleton files (the Configure script and anything in the config/ subdirectory) so that compilation is possible (or easier, or more efficient) on an operating system previously not catered for. This includes discovering and removing idiosyncrasies in the code that would hinder its portability.

The third type is to modify existing (mathematical) code, either to correct bugs, to add new functionality to existing functions, or to improve their efficiency.

Finally the last type is to add new functions to PARI. We explain here how to do this, so that in particular the new function can be called from gp.

5.8.2 Coding guidelines. Code your function in a file of its own, using as a guide other functions in the PARI sources. One important thing to remember is to clean the stack before exiting your main function, since otherwise successive calls to the function clutters the stack with unnecessary garbage, and stack overflow occurs sooner. Also, if it returns a GEN and you want it to be accessible to gp, you have to make sure this GEN is suitable for gerepileupto (see Section 4.3).

If error messages or warnings are to be generated in your function, use pari_err and pari_warn respectively. Recall that pari_err does not return but ends with a longjmp statement. As well, instead of explicit printf / fprintf statements, use the following encapsulated variants:

void pari_putc(char c): write character c to the output stream.

void pari_puts(char *s): write s to the output stream.

void pari_printf(const char *fmt, ...): write following arguments to the output stream, according to the conversion specifications in format fmt (see printf).

void err_printf(const char *fmt, ...): as pari_printf, writing to PARI's current error stream.

void err_flush(void) flush error stream.

Declare all public functions in an appropriate header file, if you want to access them from C. The other functions should be declared static in your file.

Your function is now ready to be used in library mode after compilation and creation of the library. If possible, compile it as a shared library (see the Makefile coming with the extgcd example in the distribution). It is however still inaccessible from gp.

5.8.3 GP prototypes, parser codes. A *GP prototype* is a character string describing all the GP parser needs to know about the function prototype. It contains a sequence of the following atoms:

• Return type: `GEN` by default (must be valid for `gerepileupto`), otherwise the following can appear as the *first* char of the code string:

`i`	return `int`
`l`	return `long`
`u`	return `ulong`
`v`	return `void`
`m`	return a `GEN` which is not `gerepile`-safe.

The `m` code is used for member functions, to avoid unnecessary copies. A copy opcode is generated by the compiler if the result needs to be kept safe for later use.

• Mandatory arguments, appearing in the same order as the input arguments they describe:

`G`	`GEN`
`&`	`*GEN`
`L`	`long` (we implicitly typecast `int` to `long`)
`U`	`ulong`
`V`	loop variable
`n`	variable, expects a variable number (a `long`, not an `*entree`)
`W`	a `GEN` which is a lvalue to be modified in place (for `t_LIST`)
`r`	raw input (treated as a string without quotes). Quoted args are copied as strings Stops at first unquoted `')'` or `','`. Special chars can be quoted using `'\'` Example: `aa"b\n)"c` yields the string `"aab\n)c"`
`s`	expanded string. Example: `Pi"x"2` yields `"3.142x2"` Unquoted components can be of any PARI type, converted to string following current output format
`I`	closure whose value is ignored, as in `for` loops, to be processed by `void closure_evalvoid(GEN C)`
`E`	closure whose value is used, as in `sum` loops, to be processed by `void closure_evalgen(GEN C)`
`J`	implicit function of arity 1, as in `parsum` loops, to be processed by `void closure_callgen1(GEN C)`

A *closure* is a GP function in compiled (bytecode) form. It can be efficiently evaluated using the `closure_eval`*xxx* functions.

• Automatic arguments:

`f`	Fake `*long`. C function requires a pointer but we do not use the resulting `long`
`b`	current real precision in bits
`p`	current real precision in words
`P`	series precision (default `seriesprecision`, global variable `precdl` for the library)
`C`	lexical context (internal, for `eval`, see `localvars_read_str`)

• Syntax requirements, used by functions like `for`, `sum`, etc.:

`=`	separator `=` required at this point (between two arguments)

• Optional arguments and default values:

`E*`	any number of expressions, possibly 0 (see `E`)
`s*`	any number of strings, possibly 0 (see `s`)

D*xxx* argument can be omitted and has a default value

The E* code reads all remaining arguments in closure context and passes them as a single t_VEC. The s* code reads all remaining arguments in *string context* and passes the list of strings as a single t_VEC. The automatic concatenation rules in string context are implemented so that adjacent strings are read as different arguments, as if they had been comma-separated. For instance, if the remaining argument sequence is: "xx" 1, "yy", the s* atom sends [a, b, c], where *a*, *b*, *c* are GENs of type t_STR (content "xx"), t_INT (equal to 1) and t_STR (content "yy").

The format to indicate a default value (atom starts with a D) is "D*value*,*type*,", where *type* is the code for any mandatory atom (previous group), *value* is any valid GP expression which is converted according to *type*, and the ending comma is mandatory. For instance D0,L, stands for "this optional argument is converted to a long, and is 0 by default". So if the user-given argument reads 1 + 3 at this point, 4L is sent to the function; and 0L if the argument is omitted. The following special notations are available:

- DG optional GEN, send NULL if argument omitted.
- D& optional *GEN, send NULL if argument omitted. The argument must be prefixed by &.
- DI, DE optional closure, send NULL if argument omitted.
- DP optional long, send precdl if argument omitted.
- DV optional *entree, send NULL if argument omitted.
- Dn optional variable number, −1 if omitted.
- Dr optional raw string, send NULL if argument omitted.
- Ds optional char *, send NULL if argument omitted.

Hardcoded limit. C functions using more than 20 arguments are not supported. Use vectors if you really need that many parameters.

When the function is called under gp, the prototype is scanned and each time an atom corresponding to a mandatory argument is met, a user-given argument is read (gp outputs an error message it the argument was missing). Each time an optional atom is met, a default value is inserted if the user omits the argument. The "automatic" atoms fill in the argument list transparently, supplying the current value of the corresponding variable (or a dummy pointer).

For instance, here is how you would code the following prototypes, which do not involve default values:

```
GEN f(GEN x, GEN y, long prec)   ----> "GGp"
void f(GEN x, GEN y, long prec)  ----> "vGGp"
void f(GEN x, long y, long prec) ----> "vGLp"
long f(GEN x)                    ----> "lG"
int f(long x)                    ----> "iL"
```

If you want more examples, gp gives you easy access to the parser codes attached to all GP functions: just type \h *function*. You can then compare with the C prototypes as they stand in paridecl.h.

Remark. If you need to implement complicated control statements (probably for some improved summation functions), you need to know how the parser implements closures and lexicals and how the evaluator lets you deal with them, in particular the `push_lex` and `pop_lex` functions. Check their descriptions and adapt the source code in `language/sumiter.c` and `language/intnum.c`.

5.8.4 Integration with gp as a shared module.

In this section we assume that your Operating System is supported by `install`. You have written a function in C following the guidelines is Section 5.8.2; in case the function returns a `GEN`, it must satisfy `gerepileupto` assumptions (see Section 4.3).

You then succeeded in building it as part of a shared library and want to finally tell `gp` about your function. First, find a name for it. It does not have to match the one used in library mode, but consistency is nice. It has to be a valid GP identifier, i.e. use only alphabetic characters, digits and the underscore character (_), the first character being alphabetic.

Then figure out the correct parser code corresponding to the function prototype (as explained in Section 5.8.3) and write a GP script like the following:

```
install(libname, code, gpname, library)
addhelp(gpname, "some help text")
```

The addhelp part is not mandatory, but very useful if you want others to use your module. `libname` is how the function is named in the library, usually the same name as one visible from C.

Read that file from your `gp` session, for instance from your preferences file (or `gprc`), and that's it. You can now use the new function *gpname* under `gp`, and we would very much like to hear about it!

Example. A complete description could look like this:

```
{
  install(bnfinit0, "GD0,L,DGp", ClassGroupInit, "libpari.so");
  addhelp(ClassGroupInit, "ClassGroupInit(P,{flag=0},{data=[]}):
    compute the necessary data for ...");
}
```

which means we have a function `ClassGroupInit` under `gp`, which calls the library function `bnfinit0` . The function has one mandatory argument, and possibly two more (two `'D'` in the code), plus the current real precision. More precisely, the first argument is a `GEN`, the second one is converted to a `long` using `itos` (`0` is passed if it is omitted), and the third one is also a `GEN`, but we pass `NULL` if no argument was supplied by the user. This matches the C prototype (from `paridecl.h`):

```
GEN bnfinit0(GEN P, long flag, GEN data, long prec)
```

This function is in fact coded in `basemath/buch2.c`, and is in this case completely identical to the GP function `bnfinit` but `gp` does not need to know about this, only that it can be found somewhere in the shared library `libpari.so`.

Important note. You see in this example that it is the function's responsibility to correctly interpret its operands: `data = NULL` is interpreted *by the function* as an empty vector. Note that since `NULL` is never a valid `GEN` pointer, this trick always enables you to distinguish between a default value and actual input: the user could explicitly supply an empty vector!

5.8.5 Library interface for `install`.

There is a corresponding library interface for this `install` functionality, letting you expand the GP parser/evaluator available in the library with new functions from your C source code. Functions such as `gp_read_str` may then evaluate a GP expression sequence involving calls to these new function!

```
entree * install(void *f, const char *gpname, const char *code)
```

where `f` is the (address of the) function (cast to `void*`), `gpname` is the name by which you want to access your function from within your GP expressions, and `code` is as above.

5.8.6 Integration by patching `gp`.

If `install` is not available, and installing Linux or a BSD operating system is not an option (why?), you have to hardcode your function in the `gp` binary. Here is what needs to be done:

• Fetch the complete sources of the PARI distribution.

• Drop the function source code module in an appropriate directory (a priori `src/modules`), and declare all public functions in `src/headers/paridecl.h`.

• Choose a help section and add a file `src/functions/`*section*`/`*gpname* containing the following, keeping the notation above:

```
Function:   gpname
Section:    section
C-Name:     libname
Prototype:  code
Help:       some help text
```

(If the help text does not fit on a single line, continuation lines must start by a whitespace character.) Two GP2C-related fields (`Description` and `Wrapper`) are also available to improve the code GP2C generates when compiling scripts involving your function. See the GP2C documentation for details.

• Launch `Configure`, which should pick up your C files and build an appropriate `Makefile`. At this point you can recompile `gp`, which will first rebuild the functions database.

Example. We reuse the `ClassGroupInit` / `bnfinit0` from the preceding section. Since the C source code is already part of PARI, we only need to add a file

```
functions/number_fields/ClassGroupInit
```

containing the following:

```
Function: ClassGroupInit
Section: number_fields
C-Name: bnfinit0
Prototype: GD0,L,DGp
Help: ClassGroupInit(P,{flag=0},{tech=[]}): this routine does ...
```

and recompile `gp`.

5.9 Globals related to PARI configuration.

5.9.1 PARI version numbers.

paricfg_version_code encodes in a single long, the Major and minor version numbers as well as the patchlevel.

long PARI_VERSION(long M, long m, long p) produces the version code attached to release $M.m.p$. Each code identifies a unique PARI release, and corresponds to the natural total order on the set of releases (bigger code number means more recent release).

PARI_VERSION_SHIFT is the number of bits used to store each of the integers M, m, p in the version code.

paricfg_vcsversion is a version string related to the revision control system used to handle your sources, if any. For instance git-*commit hash* if compiled from a git repository.

The two character strings paricfg_version and paricfg_buildinfo, correspond to the first two lines printed by gp just before the Copyright message. The character string paricfg_compiledate is the date of compilation which appears on the next line. The character string paricfg_mt_engine is the name of the threading engine on the next line.

In the string paricfg_buildinfo, the substring "%s" needs to be substituted by the output of the function pari_kernel_version.

const char * pari_kernel_version(void)

GEN pari_version() returns the version number as a PARI object, a t_VEC with three t_INT and one t_STR components.

5.9.2 Miscellaneous.

paricfg_datadir: character string. The location of PARI's datadir.

paricfg_gphelp: character string. The name of an external help command for ?? (such as the gphelp script)

Chapter 6:
Arithmetic kernel: Level 0 and 1

6.1 Level 0 kernel (operations on ulongs).

6.1.1 Micro-kernel. The Level 0 kernel simulates basic operations of the 68020 processor on which PARI was originally implemented. They need "global" `ulong` variables `overflow` (which will contain only 0 or 1) and `hiremainder` to function properly. A routine using one of these lowest-level functions where the description mentions either `hiremainder` or `overflow` must declare the corresponding

```
LOCAL_HIREMAINDER;  /* provides 'hiremainder' */
LOCAL_OVERFLOW;     /* provides 'overflow' */
```

in a declaration block. Variables `hiremainder` and `overflow` then become available in the enclosing block. For instance a loop over the powers of an `ulong p` protected from overflows could read

```
while (pk < lim)
{
  LOCAL_HIREMAINDER;
  ...
  pk = mulll(pk, p); if (hiremainder) break;
}
```

For most architectures, the functions mentioned below are really chunks of inlined assembler code, and the above 'global' variables are actually local register values.

`ulong addll(ulong x, ulong y)` adds `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong addllx(ulong x, ulong y)` adds `overflow` to the sum of the `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong subll(ulong x, ulong y)` subtracts `x` and `y`, returns the lower `BITS_IN_LONG` bits and put the carry (borrow) bit into `overflow`.

`ulong subllx(ulong x, ulong y)` subtracts `overflow` from the difference of `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry (borrow) bit into `overflow`.

`int bfffo(ulong x)` returns the number of leading zero bits in `x`. That is, the number of bit positions by which it would have to be shifted left until its leftmost bit first becomes equal to 1, which can be between 0 and `BITS_IN_LONG` − 1 for nonzero `x`. When `x` is 0, the result is undefined.

`ulong mulll(ulong x, ulong y)` multiplies `x` by `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong addmul(ulong x, ulong y)` adds `hiremainder` to the product of `x` and `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong divll(ulong x, ulong y)` returns the quotient of $(\texttt{hiremainder} * 2^{\texttt{BITS\_IN\_LONG}}) + \texttt{x}$ by `y` and stores the remainder into `hiremainder`. An error occurs if the quotient cannot be represented by an `ulong`, i.e. if initially $\texttt{hiremainder} \geq \texttt{y}$.

`long hammingl(ulong x))` returns the Hamming weight of x, i.e. the number of nonzero bits in its binary expansion.

Obsolete routines. Those functions are awkward and no longer used; they are only provided for backward compatibility:

`ulong shiftl(ulong x, ulong y)` returns x shifted left by y bits, i.e. x `<<` y, where we assume that $0 \leq y \leq \texttt{BITS\_IN\_LONG}$. The global variable `hiremainder` receives the bits that were shifted out, i.e. x `>>` $(\texttt{BITS\_IN\_LONG} - y)$.

`ulong shiftlr(ulong x, ulong y)` returns x shifted right by y bits, i.e. x `>>` y, where we assume that $0 \leq y \leq \texttt{BITS\_IN\_LONG}$. The global variable `hiremainder` receives the bits that were shifted out, i.e. x `<<` $(\texttt{BITS\_IN\_LONG} - y)$.

6.1.2 Modular kernel. The following routines are not part of the level 0 kernel per se, but implement modular operations on words in terms of the above. They are written so that no overflow may occur. Let $m \geq 1$ be the modulus; all operands representing classes modulo m are assumed to belong to $[0, m-1]$. The result may be wrong for a number of reasons otherwise: it may not be reduced, overflow can occur, etc.

`int odd(ulong x)` returns 1 if x is odd, and 0 otherwise.

`int both_odd(ulong x, ulong y)` returns 1 if x and y are both odd, and 0 otherwise.

`ulong invmod2BIL(ulong x)` returns the smallest positive representative of x^{-1} mod $2^{\texttt{BITS\_IN\_LONG}}$, assuming x is odd.

`ulong Fl_add(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of $x + y$ modulo m.

`ulong Fl_neg(ulong x, ulong m)` returns the smallest nonnegative representative of $-x$ modulo m.

`ulong Fl_sub(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of $x - y$ modulo m.

`long Fl_center(ulong x, ulong m, ulong mo2)` returns the representative in $]-m/2, m/2]$ of x modulo m. Assume $0 \leq x < m$ and $\texttt{mo2} = m >> 1$.

`ulong Fl_mul(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of xy modulo m.

`ulong Fl_double(ulong x, ulong m)` returns $2x$ modulo m.

`ulong Fl_triple(ulong x, ulong m)` returns $3x$ modulo m.

`ulong Fl_halve(ulong x, ulong m)` returns z such that $2z = x$ modulo m assuming such z exists.

`ulong Fl_sqr(ulong x, ulong m)` returns the smallest nonnegative representative of x^2 modulo m.

`ulong Fl_inv(ulong x, ulong m)` returns the smallest positive representative of x^{-1} modulo m. If x is not invertible mod m, raise an exception.

`ulong Fl_invsafe(ulong x, ulong m)` returns the smallest positive representative of x^{-1} modulo m. If x is not invertible mod m, return 0 (which is ambiguous if $m = 1$).

`ulong Fl_invgen(ulong x, ulong m, ulong *pg)` set `*pg` to $g = \gcd(x, m)$ and return u in $(\mathbf{Z}/m\mathbf{Z})^*$ such that $xu = g$ modulo m. We have $g = 1$ if and only if x is invertible, and in this case u is its inverse.

`ulong Fl_div(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of xy^{-1} modulo m. If y is not invertible mod m, raise an exception.

`ulong Fl_powu(ulong x, ulong n, ulong m)` returns the smallest nonnegative representative of x^n modulo m.

`GEN Fl_powers(ulong x, long n, ulong p)` returns $[\mathtt{x}^0, \dots, \mathtt{x}^\mathtt{n}]$ modulo m, as a `t_VECSMALL`.

`ulong Fl_sqrt(ulong x, ulong p)` returns the square root of `x` modulo `p` (smallest nonnegative representative). Assumes `p` to be prime, and `x` to be a square modulo `p`.

`ulong Fl_sqrtl(ulong x, ulong l, ulong p)` returns a l-the root of `x` modulo `p`. Assumes `p` to be prime and $p \equiv 1 \pmod{l}$, and `x` to be a l-th power modulo `p`.

`ulong Fl_sqrtn(ulong a, ulong n, ulong p, ulong *zn)` returns `ULONG_MAX` if a is not an n-th power residue mod p. Otherwise, returns an n-th root of a; if `zn` is not `NULL` set it to a primitive m-th root of 1, $m = \gcd(p-1, n)$ allowing to compute all m solutions in $\mathbf{F}_p$ of the equation $x^n = a$.

`ulong Fl_log(ulong a, ulong g, ulong ord, ulong p)` Let g such that $g^{ord} \equiv 1 \pmod{p}$. Return an integer e such that $a^e \equiv g \pmod{p}$. If e does not exist, the result is undefined.

`ulong Fl_order(ulong a, ulong o, ulong p)` returns the order of the `Fp a`. It is assumed that `o` is a multiple of the order of `a`, 0 being allowed (no nontrivial information).

`ulong random_Fl(ulong p)` returns a pseudo-random integer uniformly distributed in $0, 1, \dots p-1$.

`ulong nonsquare_Fl(ulong p)` return a quadratic nonresidue modulo p, assuming p is an odd prime. If p is 3 mod 4, return $p - 1$, else return the smallest (prime) nonresidue.

`ulong pgener_Fl(ulong p)` returns the smallest primitive root modulo `p`, assuming `p` is prime.

`ulong pgener_Zl(ulong p)` returns the smallest primitive root modulo p^k, $k > 1$, assuming p is an odd prime.

`ulong pgener_Fl_local(ulong p, GEN L)`, see `gener_Fp_local`, `L` is an `Flv`.

`ulong factorial_Fl(long n, ulong p)` return $n!$ mod p.

6.1.3 Modular kernel with "precomputed inverse".

This is based on an algorithm by T. Grandlund and N. Möller in "Improved division by invariant integers" `http://gmplib.org/~tege/division-paper.pdf`.

In the following, we set $B =$ `BITS_IN_LONG`.

`ulong get_Fl_red(ulong p)` returns a pseudoinverse pi for p. Namely an integer $0 < pi < B$ such that, given $0 \le x < B^2$ (by two long words), we can compute the Euclidean quotient and remainder of x modulo p by performing 2 multiplications and some additions. Precisely, once we set $q = 2^k p$ for the unique k such that $B/2 \le q < B$, the pseudoinverse pi is equal to the Euclidean quotient of $B^2 - qB + B - 1$ by q. In particular $(pi + B)/B^2$ is very close to $1/q$.

Note that this algorithm is generally less efficient than ordinary quotient and remainders (`divll` or even `/` and `%`) when $0 \leq x < B$ and $p \leq B^{1/2}$ are small. High level functions below allow setting $pi = 0$ to cater for this possibility and avoid calling `get_Fl_red` for arguments where the standard algorithm is preferable.

`ulong divll_pre(ulong x, ulong p, ulong pi)` as divll, where pi is the pseudoinverse of p.

`ulong remll_pre(ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of $u_1 2^B + u_0$ modulo p, assuming pi is the pseudoinverse of p. This function is faster if $u_1 < p$.

`ulong remlll_pre(ulong u2, ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of $u_2 2^{2B} + u_1 2^B + u_0$ modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_sqr_pre(ulong x, ulong p, ulong pi)` returns x^2 modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_mul_pre(ulong x, ulong y, ulong p, ulong pi)` returns xy modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_addmul_pre(ulong a, ulong b, ulong c, ulong p, ulong pi)` returns $a+bc$ modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_addmulmul_pre(ulong a, ulong b, ulong c, ulong d, ulong p, ulong pi)` returns $ab + cd$ modulo p, assuming pi is the pseudoinverse of p.

`ulong Fl_powu_pre(ulong x, ulong n, ulong p, ulong pi)` returns x^n modulo p, assuming pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`GEN Fl_powers_pre(ulong x, long n, ulong p, ulong pi)` returns the vector (`t_VECSMALL`) $(x^0, \ldots, x^n)$, assuming pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_log_pre(ulong a, ulong g, ulong ord, ulong p, ulong pi)` as `Fl_log`, assuming pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrt_pre(ulong x, ulong p, ulong pi)` returns a square root of x modulo p, see `Fl_sqrt`. We assume pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrtl_pre(ulong x, ulong l, ulong p, ulong pi)` returns a l-the root of `x` modulo `p`, assuming p prime, $p \equiv 1 \pmod{l}$, and `x` to be a l-th power modulo `p`. We assume pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrtn_pre(ulong x, ulong n, ulong p, ulong pi, ulong *zn)` See `Fl_sqrtn`, assuming pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_2gener_pre(ulong p, ulong pi)` return a generator of the 2-Sylow subgroup of $\mathbf{F}_p^*$, to be used in `Fl_sqrt_pre_i`. We assume pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrt_pre_i(ulong x, ulong s2, ulong p, ulong pi)` as `Fl_sqrt_pre` where `s2` is the element returned by `Fl_2gener_pre`. We assume pi is the pseudoinverse of p, or 0 in which case we either use ordinary divisions if $p < B^{1/2}$ is small and call `get_Fl_red` ourselves otherwise.

6.1.4 Switching between Fl_xxx and standard operators.

Even though the Fl_xxx routines are efficient, they are slower than ordinary long operations, using the standard +, %, etc. operators. The following macro is used to choose in a portable way the most efficient functions for given operands:

int SMALL_ULONG(ulong p) true if $2p^2 < 2^{\texttt{BITS\_IN\_LONG}}$. In that case, it is possible to use ordinary operators efficiently. If $p < 2^{\texttt{BITS\_IN\_LONG}}$, one may still use the Fl_xxx routines. Otherwise, one must use generic routines. For instance, the scalar product of the GENs x and y mod p could be computed as follows.

```
long i, l = lg(x);
if (lgefint(p) > 3)
{ /* arbitrary */
  GEN s = gen_0;
  for (i = 1; i < l; i++) s = addii(s, mulii(gel(x,i), gel(y,i)));
  return modii(s, p).
}
else
{
  ulong s = 0, pp = itou(p);
  x = ZV_to_Flv(x, pp);
  y = ZV_to_Flv(y, pp);
  if (SMALL_ULONG(pp))
  { /* very small */
    for (i = 1; i < l; i++)
    {
      s += x[i] * y[i];
      if (s & HIGHBIT) s %= pp;
    }
    s %= pp;
  }
  else
  { /* small */
    for (i = 1; i < l; i++)
      s = Fl_add(s, Fl_mul(x[i], y[i], pp), pp);
  }
  return utoi(s);
}
```

In effect, we have three versions of the same code: very small, small, and arbitrary inputs. The very small and arbitrary variants use lazy reduction and reduce only when it becomes necessary: when overflow might occur (very small), and at the very end (very small, arbitrary).

6.2 Level 1 kernel (operations on longs, integers and reals).

Note. Some functions consist of an elementary operation, immediately followed by an assignment statement. They will be introduced as in the following example:

`GEN gadd[z](GEN x, GEN y[, GEN z])` followed by the explicit description of the function

`GEN gadd(GEN x, GEN y)`

which creates its result on the stack, returning a `GEN` pointer to it, and the parts in brackets indicate that there exists also a function

`void gaddz(GEN x, GEN y, GEN z)`

which assigns its result to the pre-existing object `z`, leaving the stack unchanged. These assignment variants are kept for backward compatibility but are inefficient: don't use them.

6.2.1 Creation.

`GEN cgeti(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_INT)`.

`GEN cgetipos(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to 1.

`GEN cgetineg(long n)` allocates memory on the PARI stack for a negative `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to -1.

`GEN cgetr(long n)` allocates memory on the PARI stack for a `t_REAL` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_REAL)`.

`GEN cgetc(long n)` allocates memory on the PARI stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REAL`s of length `n`.

`GEN real_1(long prec)` create a `t_REAL` equal to 1 to `prec` words of accuracy.

`GEN real_1_bit(long bitprec)` create a `t_REAL` equal to 1 to `bitprec` bits of accuracy.

`GEN real_m1(long prec)` create a `t_REAL` equal to -1 to `prec` words of accuracy.

`GEN real_0_bit(long bit)` create a `t_REAL` equal to 0 with exponent $-$`bit`.

`GEN real_0(long prec)` is a shorthand for

```
real_0_bit( -prec2nbits(prec) )
```

`GEN int2n(long n)` creates a `t_INT` equal to `1<<n` (i.e 2^n if $n \geq 0$, and 0 otherwise).

`GEN int2u(ulong n)` creates a `t_INT` equal to 2^n.

`GEN int2um1(long n)` creates a `t_INT` equal to $2^n - 1$.

`GEN real2n(long n, long prec)` create a `t_REAL` equal to 2^n to `prec` words of accuracy.

`GEN real_m2n(long n, long prec)` create a `t_REAL` equal to -2^n to `prec` words of accuracy.

`GEN strtoi(char *s)` convert the character string `s` to a nonnegative `t_INT`. Decimal numbers, hexadecimal numbers prefixed by `0x` and binary numbers prefixed by `0b` are allowed. The string `s` consists exclusively of digits: no leading sign, no whitespace. Leading zeroes are discarded.

`GEN strtor(char *s, long prec)` convert the character string `s` to a nonnegative `t_REAL` of precision `prec`. The string `s` consists exclusively of digits and optional decimal point and exponent (`e` or `E`): no leading sign, no whitespace. Leading zeroes are discarded.

6.2.2 Assignment. In this section, the z argument in the z-functions must be of type t_INT or t_REAL.

void mpaff(GEN x, GEN z) assigns x into z (where x and z are t_INT or t_REAL). Assumes that lg(z) > 2.

void affii(GEN x, GEN z) assigns the t_INT x into the t_INT z.

void affir(GEN x, GEN z) assigns the t_INT x into the t_REAL z. Assumes that lg(z) > 2.

void affiz(GEN x, GEN z) assigns t_INT x into t_INT or t_REAL z. Assumes that lg(z) > 2.

void affsi(long s, GEN z) assigns the long s into the t_INT z. Assumes that lg(z) > 2.

void affsr(long s, GEN z) assigns the long s into the t_REAL z. Assumes that lg(z) > 2.

void affsz(long s, GEN z) assigns the long s into the t_INT or t_REAL z. Assumes that lg(z) > 2.

void affui(ulong u, GEN z) assigns the ulong u into the t_INT z. Assumes that lg(z) > 2.

void affur(ulong u, GEN z) assigns the ulong u into the t_REAL z. Assumes that lg(z) > 2.

void affrr(GEN x, GEN z) assigns the t_REAL x into the t_REAL z.

void affgr(GEN x, GEN z) assigns the scalar x into the t_REAL z, if possible.

The function affrs and affri do not exist. So don't use them.

void affrr_fixlg(GEN y, GEN z) a variant of affrr. First shorten z so that it is no longer than y, then assigns y to z. This is used in the following scenario: room is reserved for the result but, due to cancellation, fewer words of accuracy are available than had been anticipated; instead of appending meaningless 0s to the mantissa, we store what was actually computed.

Note that shortening z is not quite straightforward, since setlg(z, ly) would leave garbage on the stack, which gerepile might later inspect. It is done using

void fixlg(GEN z, long ly) see stackdummy and the examples that follow.

6.2.3 Copy.

GEN icopy(GEN x) copy relevant words of the t_INT x on the stack: the length and effective length of the copy are equal.

GEN rcopy(GEN x) copy the t_REAL x on the stack.

GEN leafcopy(GEN x) copy the leaf x on the stack (works in particular for t_INTs and t_REALs). Contrary to icopy, leafcopy preserves the original length of a t_INT. The obsolete form GEN mpcopy(GEN x) is still provided for backward compatibility.

This function also works on recursive types, copying them as if they were leaves, i.e. making a shallow copy in that case: the components of the copy point to the same data as the component of the source; see also shallowcopy.

GEN leafcopy_avma(GEN x, pari_sp av) analogous to gcopy_avma but simpler: assume x is a leaf and return a copy allocated as if initially we had avma equal to av. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) avma is just the return value (typecast to pari_sp).

GEN icopyspec(GEN x, long nx) copy the nx words x[2], ..., x[nx+1] to make up a new t_INT. Set the sign to 1.

6.2.4 Conversions.

GEN itor(GEN x, long prec) converts the t_INT x to a t_REAL of length prec and return the latter. Assumes that prec > 2.

long itos(GEN x) converts the t_INT x to a long if possible, otherwise raise an exception. We consider the conversion to be possible if and only if $|x| \leq$ LONG_MAX, i.e. $|x| < 2^{63}$ on a 64-bit architecture. Since the range is symmetric, the output of itos can safely be negated.

long itos_or_0(GEN x) converts the t_INT x to a long if possible, otherwise return 0.

int is_bigint(GEN n) true if itos(n) would give an error.

ulong itou(GEN x) converts the t_INT |x| to an ulong if possible, otherwise raise an exception. The conversion is possible if and only if lgefint$(x) \leq 3$.

long itou_or_0(GEN x) converts the t_INT |x| to an ulong if possible, otherwise return 0.

GEN stoi(long s) creates the t_INT corresponding to the long s.

GEN stor(long s, long prec) converts the long s into a t_REAL of length prec and return the latter. Assumes that prec > 2.

GEN utoi(ulong s) converts the ulong s into a t_INT and return the latter.

GEN utoipos(ulong s) converts the *nonzero* ulong s into a t_INT and return the latter.

GEN utoineg(ulong s) converts the *nonzero* ulong s into the t_INT $-s$ and return the latter.

GEN utor(ulong s, long prec) converts the ulong s into a t_REAL of length prec and return the latter. Assumes that prec > 2.

GEN rtor(GEN x, long prec) converts the t_REAL x to a t_REAL of length prec and return the latter. If prec $<$ lg(x), round properly. If prec $>$ lg(x), pad with zeroes. Assumes that prec > 2.

The following function is also available as a special case of mkintn:

GEN uu32toi(ulong a, ulong b) returns the GEN equal to $2^{32}a + b$, *assuming* that $a, b < 2^{32}$. This does not depend on sizeof(long): the behavior is as above on both 32 and 64-bit machines.

GEN uu32toineg(ulong a, ulong b) returns the GEN equal to $-(2^{32}a+b)$, *assuming* that $a, b < 2^{32}$ and that one of a or b is positive. This does not depend on sizeof(long): the behavior is as above on both 32 and 64-bit machines.

GEN uutoi(ulong a, ulong b) returns the GEN equal to $2^{\texttt{BITS\_IN\_LONG}}a + b$.

GEN uutoineg(ulong a, ulong b) returns the GEN equal to $-(2^{\texttt{BITS\_IN\_LONG}}a + b)$.

6.2.5 Integer parts. The following four functions implement the conversion from t_REAL to t_INT using standard rounding modes. Contrary to usual semantics (complement the mantissa with an infinite number of 0), they will raise an error *precision loss in truncation* if the t_REAL represents a range containing more than one integer.

GEN ceilr(GEN x) smallest integer larger or equal to the t_REAL x (i.e. the ceil function).

GEN floorr(GEN x) largest integer smaller or equal to the t_REAL x (i.e. the floor function).

GEN roundr(GEN x) rounds the t_REAL x to the nearest integer (towards $+\infty$ in case of tie).

GEN truncr(GEN x) truncates the t_REAL x (not the same as floorr if x is negative).

The following four function are analogous, but can also treat the trivial case when the argument is a t_INT:

GEN mpceil(GEN x) as ceilr except that x may be a t_INT.

GEN mpfloor(GEN x) as floorr except that x may be a t_INT.

GEN mpround(GEN x) as roundr except that x may be a t_INT.

GEN mptrunc(GEN x) as truncr except that x may be a t_INT.

GEN diviiround(GEN x, GEN y) if x and y are t_INTs, returns the quotient x/y of x and y, rounded to the nearest integer. If x/y falls exactly halfway between two consecutive integers, then it is rounded towards $+\infty$ (as for roundr).

GEN ceil_safe(GEN x), x being a real number (not necessarily a t_REAL) returns the smallest integer which is larger than any possible incarnation of x. (Recall that a t_REAL represents an interval of possible values.) Note that gceil raises an exception if the input accuracy is too low compared to its magnitude.

GEN floor_safe(GEN x), x being a real number (not necessarily a t_REAL) returns the largest integer which is smaller than any possible incarnation of x. (Recall that a t_REAL represents an interval of possible values.) Note that gfloor raises an exception if the input accuracy is too low compared to its magnitude.

GEN trunc_safe(GEN x), x being a real number (not necessarily a t_REAL) returns the integer with the largest absolute value, which is closer to 0 than any possible incarnation of x. (Recall that a t_REAL represents an interval of possible values.)

GEN roundr_safe(GEN x) rounds the t_REAL x to the nearest integer (towards $+\infty$). Complement the mantissa with an infinite number of 0 before rounding, hence never raise an exception.

6.2.6 2-adic valuations and shifts.

long vals(long s) 2-adic valuation of the long s. Returns -1 if s is equal to 0.

long vali(GEN x) 2-adic valuation of the t_INT x. Returns -1 if x is equal to 0.

GEN mpshift(GEN x, long n) shifts the t_INT or t_REAL x by n. If n is positive, this is a left shift, i.e. multiplication by 2^{n}. If n is negative, it is a right shift by $-n$, which amounts to the truncation of the quotient of x by 2^{-n}.

GEN shifti(GEN x, long n) shifts the t_INT x by n.

GEN shiftr(GEN x, long n) shifts the t_REAL x by n.

`void shiftr_inplace(GEN x, long n)` shifts the `t_REAL` x by n, in place.

`GEN trunc2nr(GEN x, long n)` given a `t_REAL` x, returns `truncr(shiftr(x,n))`, but faster, without leaving garbage on the stack and never raising a *precision loss in truncation* error. Called by `gtrunc2n`.

`GEN mantissa2nr(GEN x, long n)` given a `t_REAL` x, returns the mantissa of $x2^n$ (disregards the exponent of x). Equivalent to

```
trunc2nr(x, n-expo(x)+bit_prec(x)-1)
```

`GEN mantissa_real(GEN z, long *e)` returns the mantissa m of z, and sets `*e` to the exponent $\texttt{bit\_accuracy}(\texttt{lg}(\texttt{z})) - 1 - \texttt{expo}(z)$, so that $z = m/2^e$.

Low-level. In the following two functions, s(ource) and t(arget) need not be valid `GEN`s (in practice, they usually point to some part of a `t_REAL` mantissa): they are considered as arrays of words representing some mantissa, and we shift globally s by $n > 0$ bits, storing the result in t. We assume that $m \leq M$ and only access $s[m], s[m+1], \ldots s[M]$ (read) and likewise for t (write); we may have $s = t$ but more general overlaps are not allowed. The word f is concatenated to s to supply extra bits.

`void shift_left(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$$s[m], s[m+1], \ldots s[M], f$$

left by n bits.

`void shift_right(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$$f, s[m], s[m+1], \ldots s[M]$$

right by n bits.

6.2.7 From `t_INT` to bits or digits in base 2^k and back.

`GEN binary_zv(GEN x)` given a `t_INT` x, return a `t_VECSMALL` of bits, from most significant to least significant.

`GEN binary_2k(GEN x, long k)` given a `t_INT` x, and $k > 0$, return a `t_VEC` of digits of x in base 2^k, as `t_INT`s, from most significant to least significant.

`GEN binary_2k_nv(GEN x, long k)` given a `t_INT` x, and $0 < k <$ `BITS_IN_LONG`, return a `t_VECSMALL` of digits of x in base 2^k, as `ulong`s, from most significant to least significant.

`GEN bits_to_int(GEN x, long l)` given a vector x of l bits (as a `t_VECSMALL` or even a pointer to a part of a larger vector, so not a proper `GEN`), return the integer $\sum_{i=1}^{l} x[i]2^{l-i}$, as a `t_INT`.

`ulong bits_to_u(GEN v, long l)` same as `bits_to_int`, where $l <$ `BITS_IN_LONG`, so we can return an `ulong`.

`GEN fromdigitsu(GEN x, GEN B)` given a `t_VECSMALL` x of length l and a `t_INT` B, return the integer $\sum_{i=1}^{l} x[i]B^{i-1}$, as a `t_INT`, where the `x[i]` are seen as unsigned integers.

`GEN fromdigits_2k(GEN x, long k)` converse of `binary_2k`; given a `t_VEC` x of length l and a positive `long` k, where each $x[i]$ is a `t_INT` with $0 \leq x[i] < 2^k$, return the integer $\sum_{i=1}^{l} x[i]2^{k(l-i)}$, as a `t_INT`.

`GEN nv_fromdigits_2k(GEN x, long k)` as `fromdigits_2k`, but with x being a `t_VECSMALL` and each $x[i]$ being a `ulong` with $0 \leq x[i] < 2^{\min\{k, \texttt{BITS\_IN\_LONG}\}}$. Here k may be any positive `long`, and the $x[i]$ are regarded as k-bit integers by truncating or extending with zeroes.

6.2.8 Integer valuation. For integers x and p, such that $x \neq 0$ and $|p| > 1$, we define $v_p(x)$ to be the largest integer exponent e such that p^e divides x. If p is prime, this is the ordinary valuation of x at p.

`long Z_pvalrem(GEN x, GEN p, GEN *r)` applied to `t_INT`s `x` $\neq 0$ and `p`, $|$`p`$| > 1$, returns $e := v_p(x)$ The quotient `x/p`^{e} is returned in `*r`. If $|$`p`$|$ is a prime, `*r` is the prime-to-`p` part of `x`.

`long Z_pval(GEN x, GEN p)` as `Z_pvalrem` but only returns $v_p(x)$.

`long Z_lvalrem(GEN x, ulong p, GEN *r)` as `Z_pvalrem`, except that `p` is an `ulong` (`p` > 1).

`long Z_lvalrem_stop(GEN *x, ulong p, int *stop)` assume $x > 0$; returns $e := v_p(x)$ and replaces x by x/p^e. Set `stop` to 1 if the new value of x is $< p^2$ (and 0 otherwise). To be used when trial dividing x by successive primes: the `stop` condition is cheaply tested while testing whether p divides x (is the quotient less than p?), and allows to decide that n is prime if no prime $< p$ divides n. Not memory-clean.

`long Z_lval(GEN x, ulong p)` as `Z_pval`, except that `p` is an `ulong` (`p` > 1).

`long u_lvalrem(ulong x, ulong p, ulong *r)` as `Z_pvalrem`, except the inputs/outputs are now `ulong`s.

`long u_lvalrem_stop(ulong *n, ulong p, int *stop)` as `Z_pvalrem_stop`.

`long u_pvalrem(ulong x, GEN p, ulong *r)` as `Z_pvalrem`, except `x` and `r` are now `ulong`s.

`long u_lval(ulong x, ulong p)` as `Z_pval`, except the inputs are now `ulong`s.

`long u_pval(ulong x, GEN p)` as `Z_pval`, except `x` is now an `ulong`.

`long z_lval(long x, ulong p)` as `u_lval`, for signed `x`.

`long z_lvalrem(long x, ulong p)` as `u_lvalrem`, for signed `x`.

`long z_pval(long x, GEN p)` as `Z_pval`, except `x` is now a `long`.

`long z_pvalrem(long x, GEN p)` as `Z_pvalrem`, except `x` is now a `long`.

`long factorial_lval(ulong n, ulong p)` returns $v_p(n!)$, assuming p is prime.

The following convenience functions generalize `Z_pval` and its variants to "containers" (`ZV` and `ZX`):

`long ZV_pvalrem(GEN x, GEN p, GEN *r)` x being a `ZV` (a vector of `t_INT`s), return the min v of the valuations of its components and set `*r` to x/p^v. Infinite loop if x is the zero vector. This function is not stack clean.

`long ZV_pval(GEN x, GEN p)` as `ZV_pvalrem` but only returns the "valuation".

`int ZV_Z_dvd(GEN x, GEN p)` returns 1 if p divides all components of x and 0 otherwise. Faster than testing `ZV_pval(x,p) >= 1`.

`long ZV_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_pvalrem`, except that `p` is an `ulong` (`p` > 1). This function is not stack-clean.

`long ZV_lval(GEN x, ulong p)` as `ZV_pval`, except that `p` is an `ulong` (`p` > 1).

`long ZX_pvalrem(GEN x, GEN p, GEN *r)` as `ZV_pvalrem`, for a `ZX` x (a `t_POL` with `t_INT` coefficients). This function is not stack-clean.

`long ZX_pval(GEN x, GEN p)` as `ZV_pval` for a `ZX` x.

`long ZX_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_lvalrem`, a `ZX` x. This function is not stack-clean.

`long ZX_lval(GEN x, ulong p)` as `ZX_pval`, except that `p` is an `ulong` ($p > 1$).

6.2.9 Generic unary operators. Let "*op*" be a unary operation among

- **neg**: negation ($-x$).
- **abs**: absolute value ($|x|$).
- **sqr**: square (x^2).

The names and prototypes of the low-level functions corresponding to *op* are as follows. The result is of the same type as `x`.

`GEN` *op*`i(GEN x)` creates the result of *op* applied to the `t_INT` `x`.

`GEN` *op*`r(GEN x)` creates the result of *op* applied to the `t_REAL` `x`.

`GEN mp`*op*`(GEN x)` creates the result of *op* applied to the `t_INT` or `t_REAL` `x`.

Complete list of available functions:

`GEN absi(GEN x)`, `GEN absr(GEN x)`, `GEN mpabs(GEN x)`

`GEN negi(GEN x)`, `GEN negr(GEN x)`, `GEN mpneg(GEN x)`

`GEN sqri(GEN x)`, `GEN sqrr(GEN x)`, `GEN mpsqr(GEN x)`

`GEN absi_shallow(GEN x)` x being a `t_INT`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `negi(`x`)` otherwise.

`GEN mpabs_shallow(GEN x)` x being a `t_INT` or a `t_REAL`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `mpneg(`x`)` otherwise.

Some miscellaneous routines:

`GEN sqrs(long x)` returns x^2.

`GEN sqru(ulong x)` returns x^2.

6.2.10 Comparison operators.

`int cmpss(long s, long t)` compares the `long` `s` to the `t_long` `t`.

`int cmpuu(ulong u, ulong v)` compares the `ulong` `u` to the `t_ulong` `v`.

`long minss(long x, long y)`

`ulong minuu(ulong x, ulong y)`

`double mindd(double x, double y)` returns the `min` of x and y.

`long maxss(long x, long y)`

`ulong maxuu(ulong x, ulong y)`

`double maxdd(double x, double y)` returns the `max` of x and y.

`int mpcmp(GEN x, GEN y)` compares the `t_INT` or `t_REAL` `x` to the `t_INT` or `t_REAL` `y`. The result is the sign of $x - y$.

`int cmpii(GEN x, GEN y)` compares the `t_INT` `x` to the `t_INT` `y`.

`int cmpir(GEN x, GEN y)` compares the `t_INT` `x` to the `t_REAL` `y`.

`int cmpis(GEN x, long s)` compares the `t_INT` `x` to the `long` `s`.

`int cmpiu(GEN x, ulong s)` compares the `t_INT` `x` to the `ulong` `s`.

`int cmpsi(long s, GEN x)` compares the `long` `s` to the `t_INT` `x`.

`int cmpui(ulong s, GEN x)` compares the `ulong` `s` to the `t_INT` `x`.

`int cmpsr(long s, GEN x)` compares the `long` `s` to the `t_REAL` `x`.

`int cmpri(GEN x, GEN y)` compares the `t_REAL` `x` to the `t_INT` `y`.

`int cmprr(GEN x, GEN y)` compares the `t_REAL` `x` to the `t_REAL` `y`.

`int cmprs(GEN x, long s)` compares the `t_REAL` `x` to the `long` `s`.

`int equalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if $x = y$, 0 otherwise.

`int equalrr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is 1 if $x = y$, 0 otherwise. Equality is decided according to the following rules: all real zeroes are equal, and different from a nonzero real; two nonzero reals are equal if all their digits coincide up to the length of the shortest of the two, and the remaining words in the mantissa of the longest are all 0.

`int equalis(GEN x, long s)` compare the `t_INT` `x` and the `long` `s`. The result is 1 if $x = y$, 0 otherwise.

`int equalsi(long s, GEN x)`

`int equaliu(GEN x, ulong s)` compare the `t_INT` `x` and the `ulong` `s`. The result is 1 if $x = y$, 0 otherwise.

`int equalui(ulong s, GEN x)`

The remaining comparison operators disregard the sign of their operands

`int absequaliu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `s`. The result is 1 if $|x| = y$, 0 otherwise. This is marginally more efficient than `equalis` even when `x` is known to be nonnegative.

`int absequalui(ulong u, GEN x)`

`int abscmpiu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `u`.

`int abscmpui(ulong u, GEN x)`

`int abscmpii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is the sign of $|x| - |y|$.

`int absequalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if $|x| = |y|$, 0 otherwise.

`int abscmprr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is the sign of $|x| - |y|$.

`int absrnz_equal2n(GEN x)` tests whether a nonzero `t_REAL` `x` is equal to $\pm 2^e$ for some integer e.

`int absrnz_equal1(GEN x)` tests whether a nonzero `t_REAL` `x` is equal to ± 1.

6.2.11 Generic binary operators. The operators in this section have arguments of C-type GEN, long, and ulong, and only t_INT and t_REAL GENs are allowed. We say an argument is a real type if it is a t_REAL GEN, and an integer type otherwise. The result is always a t_REAL unless both x and y are integer types.

Let "*op*" be a binary operation among

• **add**: addition (x + y).

• **sub**: subtraction (x - y).

• **mul**: multiplication (x * y).

• **div**: division (x / y). In the case where x and y are both integer types, the result is the Euclidean quotient, where the remainder has the same sign as the dividend x. It is the ordinary division otherwise. A division-by-0 error occurs if y is equal to 0.

The last two generic operations are defined only when arguments have integer types; and the result is a t_INT:

• **rem**: remainder ("x % y"). The result is the Euclidean remainder corresponding to div, i.e. its sign is that of the dividend x.

• **mod**: true remainder (x % y). The result is the true Euclidean remainder, i.e. nonnegative and less than the absolute value of y.

Important technical note. The rules given above fixing the output type (to t_REAL unless both inputs are integer types) are subtly incompatible with the general rules obeyed by PARI's generic functions, such as gmul or gdiv for instance: the latter return a result containing as much information as could be deduced from the inputs, so it is not true that if x is a t_INT and y a t_REAL, then gmul(x,y) is always the same as mulir(x,y). The exception is $x = 0$, in that case we can deduce that the result is an exact 0, so gmul returns gen_0, while mulir returns a t_REAL 0. Specifically, the one resulting from the conversion of gen_0 to a t_REAL of precision precision(y), multiplied by y; this determines the exponent of the real 0 we obtain.

The reason for the discrepancy between the two rules is that we use the two sets of functions in different contexts: generic functions allow to write high-level code forgetting about types, letting PARI return results which are sensible and as simple as possible; type specific functions are used in kernel programming, where we do care about types and need to maintain strict consistency: it is much easier to compute the types of results when they are determined from the types of the inputs only (without taking into account further arithmetic properties, like being nonzero).

The names and prototypes of the low-level functions corresponding to *op* are as follows. In this section, the z argument in the z-functions must be of type t_INT when no r or mp appears in the argument code (no t_REAL operand is involved, only integer types), and of type t_REAL otherwise.

GEN mp*op*[z](GEN x, GEN y[, GEN z]) applies *op* to the t_INT or t_REAL x and y. The function mpdivz does not exist (its semantic would change drastically depending on the type of the z argument), and neither do mprem[z] nor mpmod[z] (specific to integers).

GEN *op*si[z](long s, GEN x[, GEN z]) applies *op* to the long s and the t_INT x. These functions always return the global constant gen_0 (not a copy) when the sign of the result is 0.

GEN *op*sr[z](long s, GEN x[, GEN z]) applies *op* to the long s and the t_REAL x.

GEN *op*ss[z](long s, long t[, GEN z]) applies *op* to the longs s and t. These functions always return the global constant gen_0 (not a copy) when the sign of the result is 0.

GEN *op*ii[z](GEN x, GEN y[, GEN z]) applies *op* to the t_INTs x and y. These functions always return the global constant gen_0 (not a copy) when the sign of the result is 0.

GEN *op*ir[z](GEN x, GEN y[, GEN z]) applies *op* to the t_INT x and the t_REAL y.

GEN *op*is[z](GEN x, long s[, GEN z]) applies *op* to the t_INT x and the long s. These functions always return the global constant gen_0 (not a copy) when the sign of the result is 0.

GEN *op*ri[z](GEN x, GEN y[, GEN z]) applies *op* to the t_REAL x and the t_INT y.

GEN *op*rr[z](GEN x, GEN y[, GEN z]) applies *op* to the t_REALs x and y.

GEN *op*rs[z](GEN x, long s[, GEN z]) applies *op* to the t_REAL x and the long s.

Some miscellaneous routines:

long expu(ulong x) assuming $x > 0$, returns the binary exponent of the real number equal to x. This is a special case of gexpo.

GEN adduu(ulong x, ulong y)

GEN addiu(GEN x, ulong y)

GEN addui(ulong x, GEN y) adds x and y.

GEN subuu(ulong x, ulong y)

GEN subiu(GEN x, ulong y)

GEN subui(ulong x, GEN y) subtracts x by y.

GEN muluu(ulong x, ulong y) multiplies x by y.

ulong umuluu_le(ulong x, ulong y, ulong n) multiplies x by y. Return xy if $xy \leq n$ and 0 otherwise (in particular if xy does not fit in an ulong).

ulong umuluu_or_0(ulong x, ulong y) multiplies x by y. Return 0 if xy does not fit in an ulong.

GEN mului(ulong x, GEN y) multiplies x by y.

GEN muluui(ulong x, ulong y, GEN z) return xyz.

GEN muliu(GEN x, ulong y) multiplies x by y.

void addumului(ulong a, ulong b, GEN x) return $a + b|X|$.

GEN addmuliu(GEN x, GEN y, ulong u) returns $x + yu$.

GEN addmulii(GEN x, GEN y, GEN z) returns $x + yz$.

GEN addmulii_inplace(GEN x, GEN y, GEN z) returns $x + yz$, but returns x itself and not a copy if $yz = 0$. Not suitable for gerepile or gerepileupto.

GEN addmuliu_inplace(GEN x, GEN y, ulong u) returns $x + yu$, but returns x itself and not a copy if $yu = 0$. Not suitable for gerepile or gerepileupto.

GEN submuliu_inplace(GEN x, GEN y, ulong u) returns $x - yu$, but returns x itself and not a copy if $yu = 0$. Not suitable for gerepile or gerepileupto.

GEN lincombii(GEN u, GEN v, GEN x, GEN y) returns $ux + vy$.

GEN mulsubii(GEN y, GEN z, GEN x) returns $yz - x$.

`GEN submulii(GEN x, GEN y, GEN z)` returns $x - yz$.

`GEN submuliu(GEN x, GEN y, ulong u)` returns $x - yu$.

`GEN mulu_interval(ulong a, ulong b)` returns $a(a+1)\cdots b$, assuming that $a \leq b$.

`GEN mulu_interval_step(ulong a, ulong b, ulong s)` returns the product of all integers in $[a, b]$ congruent to a modulo s. Assume $a \leq b$ and $s > 0$;

`GEN muls_interval(long a, long b)` returns $a(a+1)\cdots b$, assuming that $a \leq b$.

`GEN invr(GEN x)` returns the inverse of the nonzero `t_REAL` x.

`GEN truedivii(GEN x, GEN y)` returns the true Euclidean quotient (with nonnegative remainder less than $|y|$).

`GEN truedivis(GEN x, long y)` returns the true Euclidean quotient (with nonnegative remainder less than $|y|$).

`GEN truedivsi(long x, GEN y)` returns the true Euclidean quotient (with nonnegative remainder less than $|y|$).

`GEN centermodii(GEN x, GEN y, GEN y2)`, given `t_INT`s `x`, `y`, returns z congruent to `x` modulo `y`, such that $-\mathtt{y}/2 \leq z < \mathtt{y}/2$. The function requires an extra argument `y2`, such that `y2 = shifti(y, -1)`. (In most cases, `y` is constant for many reductions and `y2` need only be computed once.)

`GEN remi2n(GEN x, long n)` returns `x` mod 2^n.

`GEN addii_sign(GEN x, long sx, GEN y, long sy)` add the `t_INT`s x and y as if their signs were `sx` and `sy`.

`GEN addir_sign(GEN x, long sx, GEN y, long sy)` add the `t_INT` x and the `t_REAL` y as if their signs were `sx` and `sy`.

`GEN addrr_sign(GEN x, long sx, GEN y, long sy)` add the `t_REAL`s x and y as if their signs were `sx` and `sy`.

`GEN addsi_sign(long x, GEN y, long sy)` add x and the `t_INT` y as if its sign was `sy`.

`GEN addui_sign(ulong x, GEN y, long sy)` add x and the `t_INT` y as if its sign was `sy`.

6.2.12 Exact division and divisibility.

`GEN diviiexact(GEN x, GEN y)` returns the Euclidean quotient `x/y`, assuming `y` divides `x`. Uses Jebelean algorithm (Jebelean-Krandick bidirectional exact division is not implemented).

`GEN diviuexact(GEN x, ulong y)` returns the Euclidean quotient `x/y`, assuming `y` divides `x` and `y` is nonzero.

`GEN diviuuexact(GEN x, ulong y, ulong z)` returns the Euclidean quotient $x/(yz)$, assuming yz divides x and $yz \neq 0$.

The following routines return 1 (true) if `y` divides `x`, and 0 otherwise. All `GEN` are assumed to be `t_INT`s:

`int dvdii(GEN x, GEN y)`, `int dvdis(GEN x, long y)`, `int dvdiu(GEN x, ulong y)`,

`int dvdsi(long x, GEN y)`, `int dvdui(ulong x, GEN y)`.

The following routines return 1 (true) if `y` divides `x`, and in that case assign the quotient to `z`; otherwise they return 0. All `GEN` are assumed to be `t_INT`s:

`int dvdiiz(GEN x, GEN y, GEN z)`, `int dvdisz(GEN x, long y, GEN z)`.

`int dvdiuz(GEN x, ulong y, GEN z)` if `y` divides `x`, assigns the quotient |`x`|/`y` to `z` and returns 1 (true), otherwise returns 0 (false).

6.2.13 Division with integral operands and `t_REAL` result.

`GEN rdivii(GEN x, GEN y, long prec)`, assuming x and y are both of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec`.

`GEN rdiviiz(GEN x, GEN y, GEN z)`, assuming x and y are both of type `t_INT`, and z is a `t_REAL`, assign the quotient x/y to z.

`GEN rdivis(GEN x, long y, long prec)`, assuming `x` is of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec`.

`GEN rdivsi(long x, GEN y, long prec)`, assuming `y` is of type `t_INT`, return the quotient x/y as a `t_REAL` of precision `prec`.

`GEN rdivss(long x, long y, long prec)`, return the quotient x/y as a `t_REAL` of precision `prec`.

6.2.14 Division with remainder. The following functions return two objects, unless specifically asked for only one of them — a quotient and a remainder. The quotient is returned and the remainder is returned through the variable whose address is passed as the `r` argument. The term *true Euclidean remainder* refers to the nonnegative one (`mod`), and *Euclidean remainder* by itself to the one with the same sign as the dividend (`rem`). All `GEN`s, whether returned directly or through a pointer, are created on the stack.

`GEN dvmdii(GEN x, GEN y, GEN *r)` returns the Euclidean quotient of the `t_INT` `x` by a `t_INT` `y` and puts the remainder into `*r`. If `r` is equal to `NULL`, the remainder is not created, and if `r` is equal to `ONLY_REM`, only the remainder is created and returned. In the generic case, the remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`. The remainder is always of the sign of the dividend `x`. If the remainder is 0 set `r = gen_0`.

`void dvmdiiz(GEN x, GEN y, GEN z, GEN t)` assigns the Euclidean quotient of the `t_INT`s `x` and `y` into the `t_INT` `z`, and the Euclidean remainder into the `t_INT` `t`.

Analogous routines `dvmdis[z]`, `dvmdsi[z]`, `dvmdss[z]` are available, where `s` denotes a `long` argument. But the following routines are in general more flexible:

`long sdivss_rem(long s, long t, long *r)` computes the Euclidean quotient and remainder of the longs `s` and `t`. Puts the remainder into `*r`, and returns the quotient. The remainder is of the sign of the dividend `s`, and has strictly smaller absolute value than `t`.

`long sdivsi_rem(long s, GEN x, long *r)` computes the Euclidean quotient and remainder of the `long` `s` by the `t_INT` `x`. As `sdivss_rem` otherwise.

`long sdivsi(long s, GEN x)` as `sdivsi_rem`, without remainder.

`GEN divis_rem(GEN x, long s, long *r)` computes the Euclidean quotient and remainder of the `t_INT` `x` by the `long` `s`. As `sdivss_rem` otherwise.

`GEN absdiviu_rem(GEN x, ulong s, ulong *r)` computes the Euclidean quotient and remainder of *absolute value* of the `t_INT` `x` by the `ulong` `s`. As `sdivss_rem` otherwise.

`ulong uabsdiviu_rem(GEN n, ulong d, ulong *r)` as `absdiviu_rem`, assuming that $|n|/d$ fits into an `ulong`.

`ulong uabsdivui_rem(ulong x, GEN y, ulong *rem)` computes the Euclidean quotient and remainder of x by $|y|$. As `sdivss_rem` otherwise.

`ulong udivuu_rem(ulong x, ulong y, ulong *rem)` computes the Euclidean quotient and remainder of x by y. As `sdivss_rem` otherwise.

`ulong ceildivuu(ulong x, ulong y)` return the ceiling of x/y.

`GEN divsi_rem(long s, GEN y, long *r)` computes the Euclidean quotient and remainder of the `long s` by the `GEN y`. As `sdivss_rem` otherwise.

`GEN divss_rem(long x, long y, long *r)` computes the Euclidean quotient and remainder of the `long x` by the `long y`. As `sdivss_rem` otherwise.

`GEN truedvmdii(GEN x, GEN y, GEN *r)`, as `dvmdii` but with a nonnegative remainder.

`GEN truedvmdis(GEN x, long y, GEN *z)`, as `dvmdis` but with a nonnegative remainder.

`GEN truedvmdsi(long x, GEN y, GEN *z)`, as `dvmdsi` but with a nonnegative remainder.

6.2.15 Modulo to longs. The following variants of `modii` do not clutter the stack:

`long smodis(GEN x, long y)` computes the true Euclidean remainder of the `t_INT x` by the `long y`. This is the nonnegative remainder, not the one whose sign is the sign of `x` as in the `div` functions.

`long smodss(long x, long y)` computes the true Euclidean remainder of the `long x` by a `long y`.

`ulong umodsu(long x, ulong y)` computes the true Euclidean remainder of the `long x` by a `ulong y`.

`ulong umodiu(GEN x, ulong y)` computes the true Euclidean remainder of the `t_INT x` by the `ulong y`.

`ulong umodui(ulong x, GEN y)` computes the true Euclidean remainder of the `ulong x` by the `t_INT |y|`.

The routine `smodsi` does not exist, since it would not always be defined: for a *negative* `x`, if the quotient is ± 1, the result `x + |y|` would in general not fit into a `long`. Use either `umodui` or `modsi`.

These functions directly access the binary data and are thus much faster than the generic modulo functions:

`int mpodd(GEN x)` which is 1 if `x` is odd, and 0 otherwise.

`ulong Mod2(GEN x)`

`ulong Mod4(GEN x)`

`ulong Mod8(GEN x)`

`ulong Mod16(GEN x)`

`ulong Mod32(GEN x)`

`ulong Mod64(GEN x)` give the residue class of x modulo the corresponding power of 2.

`ulong umodi2n(GEN x, long n)` give the residue class of x modulo 2^n, $0 \leq n < BITS\_IN\_LONG$.

The following functions assume that $x \neq 0$ and in fact disregard the sign of x. There are about 10% faster than the safer variants above:

`long mod2(GEN x)`

`long mod4(GEN x)`

`long mod8(GEN x)`

`long mod16(GEN x)`

`long mod32(GEN x)`

`long mod64(GEN x)` give the residue class of $|x|$ modulo the corresponding power of 2, for *nonzero* `x`. As well,

`ulong mod2BIL(GEN x)` returns the least significant word of $|x|$, still assuming that $x \neq 0$.

6.2.16 Powering, Square root.

`GEN powii(GEN x, GEN n)`, assumes x and n are `t_INT`s and returns x^n.

`GEN powuu(ulong x, ulong n)`, returns x^n.

`GEN powiu(GEN x, ulong n)`, assumes x is a `t_INT` and returns x^n.

`GEN powis(GEN x, long n)`, assumes x is a `t_INT` and returns x^n (possibly a `t_FRAC` if $n < 0$).

`GEN powrs(GEN x, long n)`, assumes x is a `t_REAL` and returns x^n. This is considered as a sequence of `mulrr`, possibly empty: as such the result has type `t_REAL`, even if $n = 0$. Note that the generic function `gpowgs(x,0)` would return `gen_1`, see the technical note in Section 6.2.11.

`GEN powru(GEN x, ulong n)`, assumes x is a `t_REAL` and returns x^n (always a `t_REAL`, even if $n = 0$).

`GEN powersr(GEN e, long n)`. Given a `t_REAL` e, return the vector v of all e^i, $0 \leq i \leq n$, where $v[i] = e^{i-1}$.

`GEN powrshalf(GEN x, long n)`, assumes x is a `t_REAL` and returns $x^{n/2}$ (always a `t_REAL`, even if $n = 0$).

`GEN powruhalf(GEN x, ulong n)`, assumes x is a `t_REAL` and returns $x^{n/2}$ (always a `t_REAL`, even if $n = 0$).

`GEN powrfrac(GEN x, long n, long d)`, assumes x is a `t_REAL` and returns $x^{n/d}$ (always a `t_REAL`, even if $n = 0$).

`GEN powIs(long n)` returns $I^n \in \{1, I, -1, -I\}$ (`t_INT` for even n, `t_COMPLEX` otherwise).

`ulong upowuu(ulong x, ulong n)`, returns x^n when $< 2^{\texttt{BITS\_IN\_LONG}}$, and 0 otherwise (overflow).

`ulong upowers(ulong x, long n)`, returns $[1, x, \ldots, x^n]$ as a `t_VECSMALL`. Assume there is no overflow.

`GEN sqrtremi(GEN N, GEN *r)`, returns the integer square root S of the nonnegative `t_INT` `N` (rounded towards 0) and puts the remainder R into `*r`. Precisely, $N = S^2 + R$ with $0 \leq R \leq 2S$. If `r` is equal to `NULL`, the remainder is not created. In the generic case, the remainder is created after the quotient and can be disposed of individually with `cgiv(R)`. If the remainder is 0 set `R = gen_0`.

Uses a divide and conquer algorithm (discrete variant of Newton iteration) due to Paul Zimmermann ("Karatsuba Square Root", INRIA Research Report 3805 (1999)).

`GEN sqrti(GEN N)`, returns the integer square root S of the nonnegative `t_INT` `N` (rounded towards 0). This is identical to `sqrtremi(N, NULL)`.

`long logintall(GEN B, GEN y, GEN *ptq)` returns the floor e of $\log_y B$, where $B > 0$ and $y > 1$ are integers. If `ptq` is not `NULL`, set it to y^e. (Analogous to `logint0`, whithout sanity checks.)

`ulong ulogintall(ulong B, ulong y, ulong *ptq)` as `logintall` for `ulong` arguments.

`long logint(GEN B, GEN y)` returns the floor e of $\log_y B$, where $B > 0$ and $y > 1$ are integers.

`ulong ulogint(ulong B, ulong y)` as `logint` for `ulong` arguments.

`GEN vecpowuu(long N, ulong a)` return the vector of n^a, $n = 1, \ldots, N$. Not memory clean.

`GEN vecpowug(long N, GEN a, long prec)` return the vector of n^a, $n = 1, \ldots, N$, where the powers are computed at precision `prec`. Not memory clean.

6.2.17 GCD, extended GCD and LCM.

`long cgcd(long x, long y)` returns the GCD of `x` and `y`.

`ulong ugcd(ulong x, ulong y)` returns the GCD of `x` and `y`.

`ulong ugcdiu(GEN x, ulong y)` returns the GCD of `x` and `y`.

`ulong ugcdui(ulong x, GEN y)` returns the GCD of `x` and `y`.

`GEN coprimes_zv(ulong N)` return a `t_VECSMALL` T with N entries such that $T[i] = 1$ iff $(i, N) = 1$ and 0 otherwise.

`long clcm(long x, long y)` returns the LCM of `x` and `y`, provided it fits into a `long`. Silently overflows otherwise.

`ulong ulcm(ulong x, ulong y)` returns the LCM of `x` and `y`, provided it fits into an `ulong`. Silently overflows otherwise.

`GEN gcdii(GEN x, GEN y)`, returns the GCD of the `t_INT`s `x` and `y`.

`GEN lcmii(GEN x, GEN y)`, returns the LCM of the `t_INT`s `x` and `y`.

`GEN bezout(GEN a, GEN b, GEN *u, GEN *v)`, returns the GCD d of `t_INT`s `a` and `b` and sets `u`, `v` to the Bezout coefficients such that $\mathtt{au} + \mathtt{bv} = d$.

`long cbezout(long a, long b, long *u, long *v)`, returns the GCD d of `a` and `b` and sets `u`, `v` to the Bezout coefficients such that $\mathtt{au} + \mathtt{bv} = d$.

`GEN halfgcdii(GEN x, GEN y)` assuming `x` and `y` are `t_INT`s, returns a 2-components `t_VEC` $[M, V]$ where M is a 2×2 `t_MAT` and V a 2-component `t_COL`, both with `t_INT` entries, such that $M * [x, y] \; == V$ and such that if $V = [a, b]$, then $a \geq \left\lceil \sqrt{\max(|x|, |y|)} \right\rceil > b$.

`GEN ZV_extgcd(GEN A)` given a vector of n integers A, returns $[d, U]$, where d is the GCD of the $A[i]$ and U is a matrix in $\mathrm{GL}_n(\mathbf{Z})$ such that $AU = [0, \ldots, 0, D]$.

`GEN ZV_lcm(GEN v)` given a vector v of integers returns the LCM of its entries.

`GEN ZV_snf_gcd(GEN v, GEN N)` given a vector v of integers and a positive integer N, return the vector whose entries are the gcds $(v[i], N)$. Use case: if v gives the cyclic components for some Abelian group G of finite type, then this returns the structure of the finite groupe G/G^N.

6.2.18 Continued fractions and convergents.

GEN ZV_allpnqn(GEN x) given $x = [a_0, ..., a_n]$ a continued fraction from gboundcf, $n \geq 0$, return all convergents as $[P, Q]$, where $P = [p_0, \ldots, p_n]$ and $Q = [q_0, \ldots, q_n]$.

6.2.19 Pseudo-random integers. These routine return pseudo-random integers uniformly distributed in some interval. The all use the same underlying generator which can be seeded and restarted using getrand and setrand.

void setrand(GEN seed) reseeds the random number generator using the seed n. The seed is either a technical array output by getrand or a small positive integer, used to generate deterministically a suitable state array. For instance, running a randomized computation starting by setrand(1) twice will generate the exact same output.

GEN getrand(void) returns the current value of the seed used by the pseudo-random number generator random. Useful mainly for debugging purposes, to reproduce a specific chain of computations. The returned value is technical (reproduces an internal state array of type t_VECSMALL), and can only be used as an argument to setrand.

ulong pari_rand(void) returns a random $0 \leq x < 2^{\texttt{BITS\_IN\_LONG}}$.

long random_bits(long k) returns a random $0 \leq x < 2^k$. Assumes that $0 \leq k \leq$ BITS_IN_LONG.

ulong random_Fl(ulong p) returns a pseudo-random integer in $0, 1, \ldots p - 1$.

GEN randomi(GEN n) returns a random t_INT between 0 and n − 1.

GEN randomr(long prec) returns a random t_REAL in $[0, 1[$, with precision prec.

6.2.20 Modular operations. In this subsection, all GENs are t_INT.

GEN Fp_red(GEN a, GEN m) returns a modulo m (smallest nonnegative residue). (This is identical to modii).

GEN Fp_neg(GEN a, GEN m) returns −a modulo m (smallest nonnegative residue).

GEN Fp_add(GEN a, GEN b, GEN m) returns the sum of a and b modulo m (smallest nonnegative residue).

GEN Fp_sub(GEN a, GEN b, GEN m) returns the difference of a and b modulo m (smallest nonnegative residue).

GEN Fp_center(GEN a, GEN p, GEN pov2) assuming that pov2 is shifti(p,-1) and that $-p/2 < a < p$, returns the representative of a in the symmetric residue system $] - p/2, p/2]$.

GEN Fp_center_i(GEN a, GEN p, GEN pov2) internal variant of Fp_center, not gerepile-safe: when a is already in the proper interval, it is returned as is, without a copy.

GEN Fp_mul(GEN a, GEN b, GEN m) returns the product of a by b modulo m (smallest nonnegative residue).

GEN Fp_addmul(GEN x, GEN y, GEN z, GEN p) returns $x + yz$.

GEN Fp_mulu(GEN a, ulong b, GEN m) returns the product of a by b modulo m (smallest nonnegative residue).

GEN Fp_muls(GEN a, long b, GEN m) returns the product of a by b modulo m (smallest nonnegative residue).

`GEN Fp_halve(GEN x, GEN m)` returns z such that $2z = x$ modulo m assuming such z exists.

`GEN Fp_sqr(GEN a, GEN m)` returns $\mathtt{a}^2$ modulo `m` (smallest nonnegative residue).

`ulong Fp_powu(GEN x, ulong n, GEN m)` raises `x` to the `n`-th power modulo `m` (smallest nonnegative residue). Not memory-clean, but suitable for `gerepileupto`.

`ulong Fp_pows(GEN x, long n, GEN m)` raises `x` to the `n`-th power modulo `m` (smallest nonnegative residue). A negative `n` is allowed Not memory-clean, but suitable for `gerepileupto`.

`GEN Fp_pow(GEN x, GEN n, GEN m)` returns $\mathtt{x}^\mathtt{n}$ modulo `m` (smallest nonnegative residue).

`GEN Fp_pow_init(GEN x, GEN n, long k, GEN p)` Return a table `R` that can be used with `Fp_pow_table` to compute the powers of x up to n. The table is of size $2^k \log_2(n)$.

`GEN Fp_pow_table(GEN R, GEN n, GEN p)` return x^n, where R is given by `Fp_pow_init(x,m,k,p)` for some integer $m \geq n$.

`GEN Fp_powers(GEN x, long n, GEN m)` returns $[\mathtt{x}^0, \ldots, \mathtt{x}^\mathtt{n}]$ modulo `m` as a `t_VEC` (smallest nonnegative residue).

`GEN Fp_inv(GEN a, GEN m)` returns an inverse of `a` modulo `m` (smallest nonnegative residue). Raise an error if `a` is not invertible.

`GEN Fp_invsafe(GEN a, GEN m)` as `Fp_inv`, but return `NULL` if `a` is not invertible.

`GEN Fp_invgen(GEN x, GEN m, GEN *pg)` set `*pg` to $g = \gcd(x, m)$ and return u in $(\mathbf{Z}/m\mathbf{Z})^*$ such that $xu = g$ modulo m. We have $g = 1$ if and only if x is invertible, and in this case u is its inverse.

`GEN FpV_prod(GEN x, GEN p)` returns the product of the components of x.

`GEN FpV_inv(GEN x, GEN m)` x being a vector of `t_INT`s, return the vector of inverses of the $x[i]$ mod m. The routine uses Montgomery's trick, and involves a single inversion mod m, plus $3(N-1)$ multiplications for N entries. The routine is not stack-clean: $2N$ integers mod m are left on stack, besides the N in the result.

`GEN Fp_div(GEN a, GEN b, GEN m)` returns the quotient of `a` by `b` modulo `m` (smallest nonnegative residue). Raise an error if `b` is not invertible.

`GEN Fp_divu(GEN a, ulong b, GEN m)` returns the quotient of `a` by `b` modulo `m` (smallest nonnegative residue). Raise an error if `b` is not invertible.

`int invmod(GEN a, GEN m, GEN *g)`, return 1 if `a` modulo `m` is invertible, else return 0 and set $\mathtt{g} = \gcd(\mathtt{a}, \mathtt{m})$.

In the following three functions the integer parameter `ord` can be given either as a positive `t_INT` N, or as its factorization matrix *faN*, or as a pair $[N, faN]$. The parameter may be omitted by setting it to `NULL` (the value is then $p-1$).

`GEN Fp_log(GEN a, GEN g, GEN ord, GEN p)` Let g such that $g^{ord} \equiv 1 \pmod{p}$. Return an integer e such that $a^e \equiv g \pmod{p}$. If e does not exist, the result is undefined.

`GEN Fp_order(GEN a, GEN ord, GEN p)` returns the order of the `Fp a`. Assume that `ord` is a multiple of the order of `a`.

`GEN Fp_factored_order(GEN a, GEN ord, GEN p)` returns $[o, F]$, where o is the multiplicative order of the `Fp` a in $\mathbf{F}_p^*$, and F is the factorization of o. Assume that `ord` is a multiple of the order of `a`.

`int Fp_issquare(GEN x, GEN p)` returns 1 if `x` is a square modulo `p`, and 0 otherwise.

`int Fp_ispower(GEN x, GEN n, GEN p)` returns 1 if `x` is an n-th power modulo `p`, and 0 otherwise.

`GEN Fp_sqrt(GEN x, GEN p)` returns a square root of `x` modulo `p` (the smallest nonnegative residue), where `x`, `p` are `t_INT`s, and `p` is assumed to be prime. Return `NULL` if `x` is not a quadratic residue modulo `p`.

`GEN Fp_2gener(GEN p)` return a generator of the 2-Sylow subgroup of $\mathbf{F}_p^*$. To use with `Fp_sqrt_i`.

`GEN Fp_sqrt_i(GEN x, GEN s2, GEN p)` as `Fp_sqrt` where `s2` is the element returned by `Fp_2gener`.

`GEN Fp_sqrtn(GEN a, GEN n, GEN p, GEN *zn)` returns `NULL` if a is not an n-th power residue mod p. Otherwise, returns an n-th root of a; if `zn` is not `NULL` set it to a primitive m-th root of 1, $m = \gcd(p-1, n)$ allowing to compute all m solutions in $\mathbf{F}_p$ of the equation $x^n = a$.

`GEN Zn_sqrt(GEN x, GEN n)` returns one of the square roots of `x` modulo `n` (possibly not prime), where `x` is a `t_INT` and `n` is either a `t_INT` or is given by its factorization matrix. Return `NULL` if no such square root exist.

`GEN Zn_quad_roots(GEN N, GEN B, GEN C)` solves the equation $X^2 + BX + C$ modulo N. Return `NULL` if there are no solutions. Else returns $[v, M]$ where $M \mid N$ and the `FpV` v of distinct integers (reduced, implicitly modulo M) is such that x modulo N is a solution to the equation if and only if x modulo M belongs to v. If the discriminant $B^2 - 4C$ is coprime to N, we have $M = N$ but in general M can be a strict divisor of N.

`long kross(long x, long y)` returns the Kronecker symbol $(x|y)$, i.e.-1, 0 or 1. If `y` is an odd prime, this is the Legendre symbol. (Contrary to `krouu`, `kross` also supports $\texttt{y} = 0$)

`long krouu(ulong x, ulong y)` returns the Kronecker symbol $(x|y)$, i.e. -1, 0 or 1. Assumes `y` is nonzero. If `y` is an odd prime, this is the Legendre symbol.

`long krois(GEN x, long y)` returns the Kronecker symbol $(x|y)$ of `t_INT` x and `long` `y`. As `kross` otherwise.

`long kroiu(GEN x, ulong y)` returns the Kronecker symbol $(x|y)$ of `t_INT` x and nonzero `ulong` `y`. As `krouu` otherwise.

`long krosi(long x, GEN y)` returns the Kronecker symbol $(x|y)$ of `long` x and `t_INT` `y`. As `kross` otherwise.

`long kroui(ulong x, GEN y)` returns the Kronecker symbol $(x|y)$ of `long` x and `t_INT` `y`. As `kross` otherwise.

`long kronecker(GEN x, GEN y)` returns the Kronecker symbol $(x|y)$ of `t_INT`s x and `y`. As `kross` otherwise.

`GEN factorial_Fp(long n, GEN p)` return $n!$ mod p.

`GEN pgener_Fp(GEN p)` returns the smallest primitive root modulo `p`, assuming `p` is prime.

`GEN pgener_Zp(GEN p)` returns the smallest primitive root modulo p^k, $k > 1$, assuming `p` is an odd prime.

`long Zp_issquare(GEN x, GEN p)` returns 1 if the `t_INT` x is a p-adic square, 0 otherwise.

`long Zn_issquare(GEN x, GEN n)` returns 1 if `t_INT` x is a square modulo `n` (possibly not prime), where n is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise.

`long Zn_ispower(GEN x, GEN n, GEN K, GEN *py)` returns 1 if `t_INT` x is a K-th power modulo `n` (possibly not prime), where n is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise. If `py` is not `NULL`, set it to y such that $y^K = x$ modulo n.

`GEN pgener_Fp_local(GEN p, GEN L)`, `L` being a vector of primes dividing $p-1$, returns the smallest integer $x > 1$ which is a generator of the ℓ-Sylow of $\mathbf{F}_p^*$ for every ℓ in `L`. In other words, $x^{(p-1)/\ell} \neq 1$ for all such ℓ. In particular, returns `pgener_Fp(p)` if `L` contains all primes dividing $p-1$. It is not necessary, and in fact slightly inefficient, to include $\ell = 2$, since 2 is treated separately in any case, i.e. the generator obtained is never a square.

`GEN rootsof1_Fp(GEN n, GEN p)` returns a primitive n-th root modulo the prime p.

`GEN rootsof1u_Fp(ulong n, GEN p)` returns a primitive n-th root modulo the prime p.

`ulong rootsof1_Fl(ulong n, ulong p)` returns a primitive n-th root modulo the prime p.

6.2.21 Extending functions to vector inputs.

The following functions apply f to the given arguments, recursively if they are of vector / matrix type:

`GEN map_proto_G(GEN (*f)(GEN), GEN x)` For instance, if x is a `t_VEC`, return a `t_VEC` whose components are the $f(x[i])$.

`GEN map_proto_lG(long (*f)(GEN), GEN x)` As above, applying the function `stoi( f() )`.

`GEN map_proto_GL(GEN (*f)(GEN, long), GEN x, long y)`

`GEN map_proto_lGL(long (*f)(GEN, long), GEN x, long y)`

In the last function, f implements an associative binary operator, which we extend naturally to an n-ary operator f_n for any n: by convention, $f_0() = 1$, $f_1(x) = x$, and

$$f_n(x_1, \ldots, x_n) = f(f_{n-1}(x_1, \ldots, x_{n-1}), x_n)),$$

for $n \geq 2$.

`GEN gassoc_proto(GEN (*f)(GEN, GEN), GEN x, GEN y)` If y is not `NULL`, return $f(x, y)$. Otherwise, x must be of vector type, and we return the result of f applied to its components, computed using a divide-and-conquer algorithm. More precisely, return

$$f(f(x_1, \texttt{NULL}), f(x_2, \texttt{NULL})),$$

where x_1, x_2 are the two halves of x.

6.2.22 Miscellaneous arithmetic functions.

long bigomegau(ulong n) returns the number of prime divisors of $n > 0$, counted with multiplicity.

ulong coreu(ulong n), unique squarefree integer d dividing n such that n/d is a square.

ulong coreu_fact(GEN fa) same, where fa is factoru(n).

ulong corediscs(long d, ulong *pt_f), d (possibly negative) being congruent to 0 or 1 modulo 4, return the fundamental discriminant D such that $d = D * f^2$ and set *pt_f to f (if *pt_f not NULL).

GEN coredisc2_fact(GEN fa, long s, GEN *pP, GEN *pE) let d be an integer congruent to 0 or 1 mod 4. Return $D = \texttt{coredisc}(d)$ assuming that fa is the factorization of $|d|$ and $sd > 0$ (s is the sign of d). Set *pP and *pE to the factorization of the conductor f such that $d = Df^2$, where P is a t_VEC of primes and E a t_VECSMALL of exponents.

ulong coredisc2u_fact(GEN fa, long s, GEN *pP, GEN *pE) let d be an integer congruent to 0 or 1 mod 4 whose absolute value fits in an ulong. Return the absolute value of $D = \texttt{corediscs}(d)$ assuming that fa is the factorization of $|d|$ and $sd > 0$ (s is the sign of d and D). Set *pP and *pE to the factorization of the conductor f such that $d = Df^2$, where P is a t_VECSMALL of primes and E a t_VECSMALL of exponents.

ulong eulerphiu(ulong n), Euler's totient function of n.

ulong eulerphiu_fact(GEN fa) same, where fa is factoru(n).

long moebiusu(ulong n), Moebius μ-function of n.

long moebiusu_fact(GEN fa) same, where fa is factoru(n).

ulong radicalu(ulong n), product of primes dividing n.

GEN divisorsu(ulong n), returns the divisors of n in a t_VECSMALL, sorted by increasing order.

GEN divisorsu_fact(GEN fa) same, where fa is factoru(n).

GEN divisorsu_fact_factored(GEN fa) where fa is factoru(n). Return a vector $[D, F]$, where D is a a t_VECSMALL containing the divisors of u and $F[i]$ contains $\texttt{factoru}(D[i])$.

GEN divisorsu_moebius(GEN P) returns the divisors of n of the form $\prod_{p\in S}(-p)$, $S \subset P$ in a t_VECSMALL. The vector is not sorted but its first element is guaranteed to be 1. If P is factoru(n)[1], this returns the set of $\mu(d)d$ where d runs through the squarefree divisors of n.

long numdivu(ulong n), returns the number of positive divisors of $n > 0$.

long numdivu_fact(GEN fa) same, where fa is factoru(n).

long omegau(ulong n) returns the number of prime divisors of $n > 0$.

long maxomegau(ulong x) return the optimal B such that $\omega(n) \le B$ for all $n \le x$.

long maxomegaoddu(ulong x) return the optimal B such that $\omega(n) \le B$ for all odd $n \le x$.

long uissquarefree(ulong n) returns 1 if n is square-free, and 0 otherwise.

long uissquarefree_fact(GEN fa) same, where fa is factoru(n).

long uposisfundamental(ulong x) return 1 if x is a fundamental discriminant, and 0 otherwise.

`long unegisfundamental(ulong x)` return 1 if $-x$ is a fundamental discriminant, and 0 otherwise.

`long sisfundamental(long x)` return 1 if x is a fundamental discriminant, and 0 otherwise.

`int uis_357_power(ulong x, ulong *pt, ulong *mask)` as `is_357_power` for `ulong` x.

`int uis_357_powermod(ulong x, ulong *mask)` as `uis_357_power`, but only check for 3rd, 5th or 7th powers modulo $211 \times 209 \times 61 \times 203 \times 117 \times 31 \times 43 \times 71$.

`long uisprimepower(ulong n, ulong *p)` as `isprimepower`, for `ulong` n.

`int uislucaspsp(ulong n)` returns 1 if the `ulong` n fails Lucas compositeness test (it thus may be prime or composite), and 0 otherwise (proving that n is composite).

`int uis2psp(ulong n)` returns 1 if the odd `ulong` $n > 1$ fails a strong Rabin-Miller test for the base 2 (it thus may be prime or composite), and 0 otherwise (proving that n is composite).

`int uispsp(ulong a, ulong n)` returns 1 if the odd `ulong` $n > 1$ fails a strong Rabin-Miller test for the base $1 < a < n$ (it thus may be prime or composite), and 0 otherwise (proving that n is composite).

`ulong sumdigitsu(ulong n)` returns the sum of decimal digits of u.

`GEN usumdiv_fact(GEN fa)`, sum of divisors of `ulong` n, where `fa` is `factoru(n)`.

`GEN usumdivk_fact(GEN fa, ulong k)`, sum of k-th powers of divisors of `ulong` n, where `fa` is `factoru(n)`.

`GEN hilbertii(GEN x, GEN y, GEN p)`, returns the Hilbert symbol (x, y) at the prime p (`NULL` for the place at infinity); x and y are `t_INT`s.

`GEN sumdedekind(GEN h, GEN k)` returns the Dedekind sum attached to the `t_INT` h and k, $k > 0$.

`GEN sumdedekind_coprime(GEN h, GEN k)` as `sumdedekind`, except that h and k are assumed to be coprime `t_INT`s.

`GEN u_sumdedekind_coprime(long h, long k)` Let $k > 0$, $0 \leq h < k$, $(h, k) = 1$. Returns $[s_1, s_2]$ in a `t_VECSMALL`, such that $s(h, k) = (s_2 + ks_1)/(12k)$. Requires $\max(h + k/2, k) <$ `LONG_MAX` to avoid overflow, in particular $k \leq (2/3)$`LONG_MAX` is fine.

Chapter 7:
Level 2 kernel

These functions deal with modular arithmetic, linear algebra and polynomials where assumptions can be made about the types of the coefficients.

7.1 Naming scheme.

A function name is built in the following way: $A_1\_\ldots\_A_n fun$ for an operation *fun* with n arguments of class $A_1, \ldots, A_n$. A class name is given by a base ring followed by a number of code letters. Base rings are among

`Fl`: $\mathbf{Z}/l\mathbf{Z}$ where $l < 2^{\texttt{BITS\_IN\_LONG}}$ is not necessarily prime. Implemented using `ulong`s

`Fp`: $\mathbf{Z}/p\mathbf{Z}$ where p is a `t_INT`, not necessarily prime. Implemented as `t_INT`s z, preferably satisfying $0 \leq z < p$. More precisely, any `t_INT` can be used as an `Fp`, but reduced inputs are treated more efficiently. Outputs from `Fp`xxx routines are reduced.

`Fq`: $\mathbf{Z}[X]/(p, T(X))$, p a `t_INT`, T a `t_POL` with `Fp` coefficients or `NULL` (in which case no reduction modulo `T` is performed). Implemented as `t_POL`s z with `Fp` coefficients, $\deg(z) < \deg \texttt{T}$, although z a `t_INT` is allowed for elements in the prime field.

`Z`: the integers $\mathbf{Z}$, implemented as `t_INT`s.

`Zp`: the p-adic integers $\mathbf{Z}_p$, implemented as `t_INT`s, for arbitrary p

`Zl`: the p-adic integers $\mathbf{Z}_p$, implemented as `t_INT`s, for $p < 2^{\texttt{BITS\_IN\_LONG}}$

`z`: the integers $\mathbf{Z}$, implemented using (signed) `long`s.

`Q`: the rational numbers $\mathbf{Q}$, implemented as `t_INT`s and `t_FRAC`s.

`Rg`: a commutative ring, whose elements can be `gadd`-ed, `gmul`-ed, etc.

Possible letters are:

`X`: polynomial in X (`t_POL` in a fixed variable), e.g. `FpX` means $\mathbf{Z}/p\mathbf{Z}[X]$

`Y`: polynomial in $Y \neq X$. This is used to resolve ambiguities. E.g. `FpXY` means $((\mathbf{Z}/p\mathbf{Z})[X])[Y]$.

`V`: vector (`t_VEC` or `t_COL`), treated as a line vector (independently of the actual type). E.g. `ZV` means $\mathbf{Z}^k$ for some k.

`C`: vector (`t_VEC` or `t_COL`), treated as a column vector (independently of the actual type). The difference with `V` is purely semantic: if the result is a vector, it will be of type `t_COL` unless mentioned otherwise. For instance the function `ZC_add` receives two integral vectors (`t_COL` or `t_VEC`, possibly different types) of the same length and returns a `t_COL` whose entries are the sums of the input coefficients.

`M`: matrix (`t_MAT`). E.g. `QM` means a matrix with rational entries

`T`: Trees. Either a leaf or a `t_VEC` of trees.

`E`: point over an elliptic curve, represented as two-component vectors `[x,y]`, except for the represented by the one-component vector `[0]`. Not all curve models are supported.

Q: representative (t_POL) of a class in a polynomial quotient ring. E.g. an FpXQ belongs to $(\mathbf{Z}/p\mathbf{Z})[X]/(T(X))$, FpXQV means a vector of such elements, etc.

n: a polynomial representative (t_POL) for a truncated power series modulo X^n. E.g. an FpXn belongs to $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$, FpXnV means a vector of such elements, etc.

x, y, m, v, c, q: as their uppercase counterpart, but coefficient arrays are implemented using t_VECSMALLs, which coefficient understood as ulongs.

x and y (and q) are implemented by a t_VECSMALL whose first coefficient is used as a code-word and the following are the coefficients , similarly to a t_POL. This is known as a 'POLSMALL'.

m are implemented by a t_MAT whose components (columns) are t_VECSMALLs. This is known as a 'MATSMALL'.

v and c are regular t_VECSMALLs. Difference between the two is purely semantic.

Omitting the letter means the argument is a scalar in the base ring. Standard functions *fun* are

add: add

sub: subtract

mul: multiply

sqr: square

div: divide (Euclidean quotient)

rem: Euclidean remainder

divrem: return Euclidean quotient, store remainder in a pointer argument. Three special values of that pointer argument modify the default behavior: NULL (do not store the remainder, used to implement div), ONLY_REM (return the remainder, used to implement rem), ONLY_DIVIDES (return the quotient if the division is exact, and NULL otherwise).

gcd: GCD

extgcd: return GCD, store Bezout coefficients in pointer arguments

pow: exponentiate

eval: evaluation / composition

7.2 Coefficient ring.

`long Rg_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the "natural" base ring over which the object x is defined.

Raise an error if it detects consistency problems in modular objects: incompatible rings (e.g. $\mathbf{F}_p$ and $\mathbf{F}_q$ for primes $p \neq q$, $\mathbf{F}_p[X]/(T)$ and $\mathbf{F}_p[X]/(U)$ for $T \neq U$). Minor discrepancies are supported if they make general sense (e.g. $\mathbf{F}_p$ and $\mathbf{F}_{p^k}$, but not $\mathbf{F}_p$ and $\mathbf{Q}_p$); `t_FFELT` and `t_POLMOD` of `t_INTMOD`s are considered inconsistent, even if they define the same field: if you need to use simultaneously these different finite field implementations, multiply the polynomial by a `t_FFELT` equal to 1 first.

• 0: none of the others (presumably multivariate, possibly inconsistent).

• `t_INT`: defined over $\mathbf{Z}$.

• `t_FRAC`: defined over $\mathbf{Q}$.

• `t_INTMOD`: defined over $\mathbf{Z}/p\mathbf{Z}$, where `*ptp` is set to p. It is not checked whether p is prime.

• `t_COMPLEX`: defined over $\mathbf{C}$ (at least one `t_COMPLEX` with at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.

• `t_REAL`: defined over $\mathbf{R}$ (at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.

• `t_PADIC`: defined over $\mathbf{Q}_p$, where `*ptp` is set to p and `*ptprec` to the p-adic accuracy.

• `t_FFELT`: defined over a finite field $\mathbf{F}_{p^k}$, where `*ptp` is set to the field characteristic p and `*ptpol` is set to a `t_FFELT` belonging to the field.

• `t_POL`: defined over a polynomial ring.

• other values are composite corresponding to quotients $R[X]/(T)$, with one primary type `t1`, describing the form of the quotient, and a secondary type `t2`, describing R. If `t` is the `RgX_type`, `t1` and `t2` are recovered using

`void RgX_type_decode(long t, long *t1, long *t2)`

`t1` is one of

`t_POLMOD`: at least one `t_POLMOD` component, set `*ppol` to the modulus,

`t_QUAD`: no `t_POLMOD`, at least one `t_QUAD` component, set `*ppol` to the modulus (`-.pol`) of the `t_QUAD`,

`t_COMPLEX`: no `t_POLMOD` or `t_QUAD`, at least one `t_COMPLEX` component, set `*ppol` to $y^2 + 1$.

and the underlying base ring R is given by `t2`, which is one of `t_INT`, `t_INTMOD` (set `*ptp`) or `t_PADIC` (set `*ptp` and `*ptprec`), with the same meaning as above.

`int RgX_type_is_composite(long t)` t as returned by `RgX_type`, return 1 if t is a composite type, and 0 otherwise.

`GEN Rg_get_0(GEN x)` returns 0 in the base ring over which x is defined, to the proper accuracy (e.g. `0`, `Mod(0,3)`, `O(5^10)`).

`GEN Rg_get_1(GEN x)` returns 1 in the base ring over which x is defined, to the proper accuracy (e.g. `0`, `Mod(0,3)`, long RgX_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the polynomial x is defined, otherwise as Rg_type.

long RgX_Rg_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the polynomial x and the element y are defined, otherwise as Rg_type.

long RgX_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the polynomials x and y are defined, otherwise as Rg_type.

long RgX_type3(GEN x, GEN y, GNE z, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the polynomials x, y and z are defined, otherwise as Rg_type.

long RgM_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the matrix x is defined, otherwise as Rg_type.

long RgM_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the matrices x and y are defined, otherwise as Rg_type.

long RgV_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the vector x is defined, otherwise as Rg_type.

long RgV_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the vectors x and y are defined, otherwise as Rg_type.

long RgM_RgC_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec) returns the "natural" base ring over which the matrix x and the vector y are defined, otherwise as Rg_type.

7.3 Modular arithmetic.

These routines implement univariate polynomial arithmetic and linear algebra over finite fields, in fact over finite rings of the form $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$, where p is not necessarily prime and $T \in (\mathbf{Z}/p\mathbf{Z})[X]$ is possibly reducible; and finite extensions thereof. All this can be emulated with t_INTMOD and t_POLMOD coefficients and using generic routines, at a considerable loss of efficiency. Also, specialized routines are available that have no obvious generic equivalent.

7.3.1 FpC / FpV, FpM. A ZV (resp. a ZM) is a t_VEC or t_COL (resp. t_MAT) with t_INT coefficients. An FpV or FpM, with respect to a given t_INT p, is the same with Fp coordinates; operations are understood over $\mathbf{Z}/p\mathbf{Z}$.

7.3.1.1 Conversions.

int Rg_is_Fp(GEN z, GEN *p), checks if z can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a t_INT or a t_INTMOD whose modulus is equal to *p, (if *p not NULL), in that case return 1, else 0. If a modulus is found it is put in *p, else *p is left unchanged.

int RgV_is_FpV(GEN z, GEN *p), z a t_VEC (resp. t_COL), checks if it can be mapped to a FpV (resp. FpC), by checking Rg_is_Fp coefficientwise.

int RgM_is_FpM(GEN z, GEN *p), z a t_MAT, checks if it can be mapped to a FpM, by checking RgV_is_FpV columnwise.

GEN Rg_to_Fp(GEN z, GEN p), z a scalar which can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a t_INT, a t_INTMOD whose modulus is divisible by p, a t_FRAC whose denominator is coprime to p, or a t_PADIC with underlying prime ℓ satisfying $p = \ell^n$ for some n (less than the accuracy of the input). Returns lift(z * Mod(1,p)), normalized.

GEN padic_to_Fp(GEN x, GEN p) special case of Rg_to_Fp, for a x a t_PADIC.

GEN RgV_to_FpV(GEN z, GEN p), z a t_VEC or t_COL, returns the FpV (as a t_VEC) obtained by applying Rg_to_Fp coefficientwise.

GEN RgC_to_FpC(GEN z, GEN p), z a t_VEC or t_COL, returns the FpC (as a t_COL) obtained by applying Rg_to_Fp coefficientwise.

GEN RgM_to_FpM(GEN z, GEN p), z a t_MAT, returns the FpM obtained by applying RgC_to_FpC columnwise.

GEN RgM_Fp_init(GEN z, GEN p, ulong *pp), given an RgM z, whose entries can be mapped to $\mathbf{F}_p$ (as per Rg_to_Fp), and a prime number p. This routine returns a normal form of z: either an F2m ($p = 2$), an Flm (p fits into an ulong) or an FpM. In the first two cases, pp is set to itou(p), and to 0 in the last.

The functions above are generally used as follows:

```
GEN add(GEN x, GEN y)
{
  GEN p = NULL;
  if (Rg_is_Fp(x, &p) && Rg_is_Fp(y, &p) && p)
  {
    x = Rg_to_Fp(x, p); y = Rg_to_Fp(y, p);
    z = Fp_add(x, y, p);
    return Fp_to_mod(z);
  }
  else return gadd(x, y);
}
```

GEN FpC_red(GEN z, GEN p), z a ZC. Returns lift(Col(z) * Mod(1,p)), hence a t_COL.

GEN FpV_red(GEN z, GEN p), z a ZV. Returns lift(Vec(z) * Mod(1,p)), hence a t_VEC

GEN FpM_red(GEN z, GEN p), z a ZM. Returns lift(z * Mod(1,p)), which is an FpM.

7.3.1.2 Basic operations.

GEN random_FpC(long n, GEN p) returns a random FpC with n components.

GEN random_FpV(long n, GEN p) returns a random FpV with n components.

GEN FpC_center(GEN z, GEN p, GEN pov2) returns a t_COL whose entries are the Fp_center of the gel(z,i).

GEN FpM_center(GEN z, GEN p, GEN pov2) returns a matrix whose entries are the Fp_center of the gcoeff(z,i,j).

void FpC_center_inplace(GEN z, GEN p, GEN pov2) in-place version of FpC_center, using affii.

void FpM_center_inplace(GEN z, GEN p, GEN pov2) in-place version of FpM_center, using affii.

GEN FpC_add(GEN x, GEN y, GEN p) adds the ZC x and y and reduce modulo p to obtain an FpC.

GEN FpV_add(GEN x, GEN y, GEN p) same as FpC_add, returning and FpV.

`GEN FpM_add(GEN x, GEN y, GEN p)` adds the two `ZM`s `x` and `y` (assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpC_sub(GEN x, GEN y, GEN p)` subtracts the `ZC` y to the `ZC` x and reduce modulo p to obtain an `FpC`.

`GEN FpV_sub(GEN x, GEN y, GEN p)` same as `FpC_sub`, returning and `FpV`.

`GEN FpM_sub(GEN x, GEN y, GEN p)` subtracts the two `ZM`s `x` and `y` (assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpC_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `ZC` `x` (seen as a column vector) by the `t_INT` `y` and reduce modulo `p` to obtain an `FpC`.

`GEN FpM_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `ZM` `x` (seen as a column vector) by the `t_INT` `y` and reduce modulo `p` to obtain an `FpM`.

`GEN FpC_FpV_mul(GEN x, GEN y, GEN p)` multiplies the `ZC` `x` (seen as a column vector) by the `ZV` `y` (seen as a row vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpM_mul(GEN x, GEN y, GEN p)` multiplies the two `ZM`s `x` and `y` (assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpM`.

`GEN FpM_powu(GEN x, ulong n, GEN p)` computes x^n where x is a square `FpM`.

`GEN FpM_FpC_mul(GEN x, GEN y, GEN p)` multiplies the `ZM` `x` by the `ZC` `y` (seen as a column vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `FpC`.

`GEN FpM_FpC_mul_FpX(GEN x, GEN y, GEN p, long v)` is a memory-clean version of

```
GEN tmp = FpM_FpC_mul(x,y,p);
return RgV_to_RgX(tmp, v);
```

`GEN FpV_FpC_mul(GEN x, GEN y, GEN p)` multiplies the `ZV` `x` (seen as a row vector) by the `ZC` `y` (seen as a column vector, assumed to have compatible dimensions), and reduce modulo `p` to obtain an `Fp`.

`GEN FpV_dotproduct(GEN x, GEN y, GEN p)` scalar product of x and y (assumed to have the same length).

`GEN FpV_dotsquare(GEN x, GEN p)` scalar product of x with itself. has `t_INT` entries.

`GEN FpV_factorback(GEN L, GEN e, GEN p)` given an `FpV` L and a `ZV` or `zv` e of the same length, return $\prod_i L_i^{e_i}$ modulo p.

7.3.1.3 `Fp`-linear algebra. The implementations are not asymptotically efficient ($O(n^3)$ standard algorithms).

`GEN FpM_deplin(GEN x, GEN p)` returns a nontrivial kernel vector, or `NULL` if none exist.

`GEN FpM_det(GEN x, GEN p)` as `det`

`GEN FpM_gauss(GEN a, GEN b, GEN p)` as `gauss`, where a and b are `FpM`.

`GEN FpM_FpC_gauss(GEN a, GEN b, GEN p)` as `gauss`, where a is a `FpM` and b a `FpC`.

`GEN FpM_image(GEN x, GEN p)` as `image`

`GEN FpM_intersect(GEN x, GEN y, GEN p)` as `intersect`

`GEN FpM_intersect_i(GEN x, GEN y, GEN p)` internal variant of `FpM_intersect` but the result is only a generating set, not necessarily an $\mathbf{F}_p$-basis. It is not `gerepile`-clean either, but suitable for `gerepileupto`.

`GEN FpM_inv(GEN x, GEN p)` returns a left inverse of `x` (the inverse if x is square), or `NULL` if `x` is not invertible.

`GEN FpM_FpC_invimage(GEN A, GEN y, GEN p)` given an `FpM` A and an `FpC` y, returns an x such that $Ax = y$, or `NULL` if no such vector exist.

`GEN FpM_invimage(GEN A, GEN y, GEN p)` given two `FpM` A and y, returns x such that $Ax = y$, or `NULL` if no such matrix exist.

`GEN FpM_ker(GEN x, GEN p)` as `ker`

`long FpM_rank(GEN x, GEN p)` as `rank`

`GEN FpM_indexrank(GEN x, GEN p)` as `indexrank`

`GEN FpM_suppl(GEN x, GEN p)` as `suppl`

`GEN FpM_hess(GEN x, GEN p)` upper Hessenberg form of x over $\mathbf{F}_p$.

`GEN FpM_charpoly(GEN x, GEN p)` characteristic polynomial of x.

7.3.1.4 `FqC`, `FqM` and `Fq`-linear algebra.

An `FqM` (resp. `FqC`) is a matrix (resp a `t_COL`) with `Fq` coefficients (with respect to given `T`, `p`), not necessarily reduced (i.e arbitrary `t_INT`s and `ZX`s in the same variable as `T`).

`GEN RgC_to_FqC(GEN z, GEN T, GEN p)`

`GEN RgM_to_FqM(GEN z, GEN T, GEN p)`

`GEN FqC_add(GEN a, GEN b, GEN T, GEN p)`

`GEN FqC_sub(GEN a, GEN b, GEN T, GEN p)`

`GEN FqC_Fq_mul(GEN a, GEN b, GEN T, GEN p)`

`GEN FqC_FqV_mul(GEN a, GEN b, GEN T, GEN p)`

`GEN FqM_FqC_gauss(GEN a, GEN b, GEN T, GEN p)` as `gauss`, where b is a `FqC`.

`GEN FqM_FqC_invimage(GEN a, GEN b, GEN T, GEN p)`

`GEN FqM_FqC_mul(GEN a, GEN b, GEN T, GEN p)`

`GEN FqM_deplin(GEN x, GEN T, GEN p)` returns a nontrivial kernel vector, or `NULL` if none exist.

`GEN FqM_det(GEN x, GEN T, GEN p)` as `det`

`GEN FqM_gauss(GEN a, GEN b, GEN T, GEN p)` as `gauss`, where b is a `FqM`.

`GEN FqM_image(GEN x, GEN T, GEN p)` as `image`

`GEN FqM_indexrank(GEN x, GEN T, GEN p)` as `indexrank`

`GEN FqM_inv(GEN x, GEN T, GEN p)` returns the inverse of `x`, or `NULL` if `x` is not invertible.

`GEN FqM_invimage(GEN a, GEN b, GEN T, GEN p)` as `invimage`

`GEN FqM_ker(GEN x, GEN T, GEN p)` as `ker`

`GEN FqM_mul(GEN a, GEN b, GEN T, GEN p)`

`long FqM_rank(GEN x, GEN T, GEN p)` as `rank`

`GEN FqM_suppl(GEN x, GEN T, GEN p)` as `suppl`

7.3.2 `Flc / Flv`, `Flm`. See `FpV`, `FpM` operations.

`GEN Flv_copy(GEN x)` returns a copy of `x`.

`GEN Flv_center(GEN z, ulong p, ulong ps2)`

`GEN random_Flv(long n, ulong p)` returns a random `Flv` with n components.

`GEN Flm_copy(GEN x)` returns a copy of `x`.

`GEN matid_Flm(long n)` returns an `Flm` which is an $n \times n$ identity matrix.

`GEN scalar_Flm(long s, long n)` returns an `Flm` which is s times the $n \times n$ identity matrix.

`GEN Flm_center(GEN z, ulong p, ulong ps2)`

`GEN Flm_Fl_add(GEN x, ulong y, ulong p)` returns $x + y * \mathrm{Id}$ (x must be square).

`GEN Flm_Fl_sub(GEN x, ulong y, ulong p)` returns $x - y * \mathrm{Id}$ (x must be square).

`GEN Flm_Flc_mul(GEN x, GEN y, ulong p)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_Flc_mul_pre(GEN x, GEN y, ulong p, ulong pi)` multiplies `x` and `y` (assumed to have compatible dimensions), assuming pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flc_Flv_mul(GEN x, GEN y, ulong p)` multiplies the column vector x by the row vector y. The result is a matrix.

`GEN Flm_Flc_mul_pre_Flx(GEN x, GEN y, ulong p, ulong pi, long sv)` return `Flv_to_Flx(Flm_Flc_mul_pre(x, y, p, pi), sv)`.

`GEN Flm_Fl_mul(GEN x, ulong y, ulong p)` multiplies the `Flm` `x` by `y`.

`GEN Flm_Fl_mul_pre(GEN x, ulong y, ulong p, ulong pi)` multiplies the `Flm` `x` by `y` assuming pi is the pseudoinverse of p, or 0 in which case we assume $p < B^{1/2}$ is small.

`GEN Flm_neg(GEN x, ulong p)` negates the `Flm` `x`.

`void Flm_Fl_mul_inplace(GEN x, ulong y, ulong p)` replaces the `Flm` `x` by `x` $*$ `y`.

`GEN Flv_Fl_mul(GEN x, ulong y, ulong p)` multiplies the `Flv` `x` by `y`.

`void Flv_Fl_mul_inplace(GEN x, ulong y, ulong p)` replaces the `Flc` `x` by `x` $*$ `y`.

`void Flv_Fl_mul_part_inplace(GEN x, ulong y, ulong p, long l)` multiplies $x[1..l]$ by y modulo p. In place.

`GEN Flv_Fl_div(GEN x, ulong y, ulong p)` divides the `Flv` `x` by `y`.

`void Flv_Fl_div_inplace(GEN x, ulong y, ulong p)` replaces the `Flv` `x` by `x/y`.

`void Flc_lincomb1_inplace(GEN X, GEN Y, ulong v, ulong q)` sets $X \leftarrow X + vY$, where X, Y are `Flc`. Memory efficient (e.g. no-op if $v = 0$), and gerepile-safe.

`GEN Flv_add(GEN x, GEN y, ulong p)` adds two `Flv`.

`void Flv_add_inplace(GEN x, GEN y, ulong p)` replaces x by $x+y$.

`GEN Flv_neg(GEN x, ulong p)` returns $-x$.

`void Flv_neg_inplace(GEN x, ulong p)` replaces x by $-x$.

`GEN Flv_sub(GEN x, GEN y, ulong p)` subtracts `y` to `x`.

`void Flv_sub_inplace(GEN x, GEN y, ulong p)` replaces x by $x-y$.

`ulong Flv_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of `x` and `y`

`ulong Flv_dotproduct_pre(GEN x, GEN y, ulong p, ulong pi)` returns the scalar product of `x` and `y` assuming pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flv_factorback(GEN L, GEN e, ulong p)` given an `Flv` L and a `zv` e of the same length, return $\prod_i L_i^{e_i}$ modulo p.

`ulong Flv_sum(GEN x, ulong p)` returns the sum of the components of x.

`ulong Flv_prod(GEN x, ulong p)` returns the product of the components of x.

`ulong Flv_prod_pre(GEN x, ulong p, ulong pi)` as `Flv_prod` assuming pi is the pseudoinverse of p.

`GEN Flv_inv(GEN x, ulong p)` returns the vector of inverses of the elements of x (as a `Flv`). Use Montgomery's trick.

`void Flv_inv_inplace(GEN x, ulong p)` in place variant of `Flv_inv`.

`GEN Flv_inv_pre(GEN x, ulong p, ulong pi)` as `Flv_inv` assuming pi is the pseudoinverse of p.

`void Flv_inv_pre_inplace(GEN x, ulong p, ulong pi)` in place variant of `Flv_inv`.

`GEN Flc_FpV_mul(GEN x, GEN y, GEN p)` multiplies x (seen as a column vector) by y (seen as a row vector, assumed to have compatible dimensions) to obtain an `Flm`.

`GEN zero_Flm(long m, long n)` creates a `Flm` with `m` x `n` components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

`GEN zero_Flm_copy(long m, long n)` creates a `Flm` with `m` x `n` components set to 0.

`GEN zero_Flv(long n)` creates a `Flv` with `n` components set to 0.

`GEN Flm_row(GEN A, long x0)` return $A[i,]$, the i-th row of the `Flm` A.

`GEN Flm_add(GEN x, GEN y, ulong p)` adds `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_sub(GEN x, GEN y, ulong p)` subtracts `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_mul(GEN x, GEN y, ulong p)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN Flm_mul_pre(GEN x, GEN y, ulong p, ulong pi)` multiplies `x` and `y` (assumed to have compatible dimensions), assuming pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flm_powers(GEN x, ulong n, ulong p)` returns $[\mathrm{x}^0, \dots, \mathrm{x}^\mathrm{n}]$ as a `t_VEC` of `Flm`s.

`GEN Flm_powu(GEN x, ulong n, ulong p)` computes x^n where x is a square `Flm`.

`GEN Flm_charpoly(GEN x, ulong p)` return the characteristic polynomial of the square `Flm` x, as a `Flx`.

`GEN Flm_deplin(GEN x, ulong p)`

`ulong Flm_det(GEN x, ulong p)`

`ulong Flm_det_sp(GEN x, ulong p)`, as `Flm_det`, in place (destroys `x`).

`GEN Flm_gauss(GEN a, GEN b, ulong p)` as `gauss`, where b is a `Flm`.

`GEN Flm_Flc_gauss(GEN a, GEN b, ulong p)` as `gauss`, where b is a `Flc`.

`GEN Flm_indexrank(GEN x, ulong p)`

`GEN Flm_inv(GEN x, ulong p)`

`GEN Flm_adjoint(GEN x, ulong p)` as `matadjoint`.

`GEN Flm_Flc_invimage(GEN A, GEN y, ulong p)` given an `Flm` A and an `Flc` y, returns an x such that $Ax = y$, or `NULL` if no such vector exist.

`GEN Flm_invimage(GEN A, GEN y, ulong p)` given two `Flm` A and y, returns x such that $Ax = y$, or `NULL` if no such matrix exist.

`GEN Flm_ker(GEN x, ulong p)`

`GEN Flm_ker_sp(GEN x, ulong p, long deplin)`, as `Flm_ker` (if `deplin=0`) or `Flm_deplin` (if `deplin=1`) , in place (destroys `x`).

`long Flm_rank(GEN x, ulong p)`

`long Flm_suppl(GEN x, ulong p)`

`GEN Flm_image(GEN x, ulong p)`

`GEN Flm_intersect(GEN x, GEN y, ulong p)`

`GEN Flm_intersect_i(GEN x, GEN y, GEN p)` internal variant of `Flm_intersect` but the result is only a generating set, not necessarily an $\mathbf{F}_p$-basis. It *is* a basis if both x and y have independent columns. It is not `gerepile`-clean either, but suitable for `gerepileupto`.

`GEN Flm_transpose(GEN x)`

`GEN Flm_hess(GEN x, ulong p)` upper Hessenberg form of x over $\mathbf{F}_p$.

7.3.3 `F2c / F2v, F2m`. An `F2v v` is a `t_VECSMALL` representing a vector over $\mathbf{F}_2$. Specifically `z[0]` is the usual codeword, `z[1]` is the number of components of v and the coefficients are given by the bits of remaining words by increasing indices.

`ulong F2v_coeff(GEN x, long i)` returns the coefficient $i \geq 1$ of x.

`void F2v_clear(GEN x, long i)` sets the coefficient $i \geq 1$ of x to 0.

`int F2v_equal0(GEN x)` returns 1 if all entries are 0, and return 0 otherwise.

`void F2v_flip(GEN x, long i)` adds 1 to the coefficient $i \geq 1$ of x.

`void F2v_set(GEN x, long i)` sets the coefficient $i \geq 1$ of x to 1.

`void F2v_copy(GEN x)` returns a copy of x.

`GEN F2v_slice(GEN x, long a, long b)` returns the `F2v` with entries $x[a]$, ..., $x[b]$. Assumes $a \leq b$.

`ulong F2m_coeff(GEN x, long i, long j)` returns the coefficient (i, j) of x.

`void F2m_clear(GEN x, long i, long j)` sets the coefficient (i, j) of x to 0.

`void F2m_flip(GEN x, long i, long j)` adds 1 to the coefficient (i, j) of x.

`void F2m_set(GEN x, long i, long j)` sets the coefficient (i, j) of x to 1.

`GEN F2m_copy(GEN x)` returns a copy of x.

`GEN F2m_transpose(GEN x)` returns the transpose of x.

`GEN F2m_row(GEN x, long j)` returns the `F2v` which corresponds to the j-th row of the `F2m` x.

`GEN F2m_rowslice(GEN x, long a, long b)` returns the `F2m` built from the a-th to b-th rows of the `F2m` x. Assumes $a \leq b$.

`GEN F2m_F2c_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN F2m_image(GEN x)` gives a subset of the columns of x that generate the image of x.

`GEN F2m_invimage(GEN A, GEN B)`

`GEN F2m_F2c_invimage(GEN A, GEN y)`

`GEN F2m_gauss(GEN a, GEN b)` as `gauss`, where b is a `F2m`.

`GEN F2m_F2c_gauss(GEN a, GEN b)` as `gauss`, where b is a `F2c`.

`GEN F2m_indexrank(GEN x)` x being a matrix of rank r, returns a vector with two `t_VECSMALL` components y and z of length r giving a list of rows and columns respectively (starting from 1) such that the extracted matrix obtained from these two vectors using `vecextract`(x, y, z) is invertible.

`GEN F2m_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN F2m_powu(GEN x, ulong n)` computes x^n where x is a square `F2m`.

`long F2m_rank(GEN x)` as `rank`.

`long F2m_suppl(GEN x)` as `suppl`.

`GEN matid_F2m(long n)` returns an `F2m` which is an $n \times n$ identity matrix.

`GEN zero_F2v(long n)` creates a `F2v` with `n` components set to 0.

`GEN const_F2v(long n)` creates a `F2v` with `n` components set to 1.

`GEN F2v_ei(long n, long i)` creates a `F2v` with `n` components set to 0, but for the i-th one, which is set to 1 (i-th vector in the canonical basis).

`GEN zero_F2m(long m, long n)` creates a `Flm` with `m` x `n` components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

`GEN zero_F2m_copy(long m, long n)` creates a `F2m` with `m` x `n` components set to 0.

`GEN F2v_to_Flv(GEN x)`

`GEN F2c_to_ZC(GEN x)`

`GEN ZV_to_F2v(GEN x)`

`GEN RgV_to_F2v(GEN x)`

`GEN F2m_to_Flm(GEN x)`

`GEN F2m_to_ZM(GEN x)`

`GEN Flv_to_F2v(GEN x)`

`GEN Flm_to_F2m(GEN x)`

`GEN ZM_to_F2m(GEN x)`

`GEN RgM_to_F2m(GEN x)`

`void F2v_add_inplace(GEN x, GEN y)` replaces x by $x+y$. It is allowed for y to be shorter than x.

`void F2v_and_inplace(GEN x, GEN y)` replaces x by the term-by term product of x and y (which is the logical `and`). It is allowed for y to be shorter than x.

`void F2v_negimply_inplace(GEN x, GEN y)` replaces x by the term-by term logical `and not` of x and y. It is allowed for y to be shorter than x.

`void F2v_or_inplace(GEN x, GEN y)` replaces x by the term-by term logical `or` of x and y. It is allowed for y to be shorter than x.

`int F2v_subset(GEN x, GEN y)` return 1 if the set of indices of non-zero components of y is a subset of the set of indices of non-zero components of x, 0 otherwise.

`ulong F2v_hamming(GEN x)` returns the Hamming weight of `x`, that is the number of nonzero entries.

`ulong F2m_det(GEN x)`

`ulong F2m_det_sp(GEN x)`, as `F2m_det`, in place (destroys `x`).

`GEN F2m_deplin(GEN x)`

`ulong F2v_dotproduct(GEN x, GEN y)` returns the scalar product of `x` and `y`

`GEN F2m_inv(GEN x)`

`GEN F2m_ker(GEN x)`

`GEN F2m_ker_sp(GEN x, long deplin)`, as `F2m_ker` (if `deplin=0`) or `F2m_deplin` (if `deplin=1`), in place (destroys `x`).

7.3.4 `F3c / F3v`, `F3m`. An `F3v v` is a `t_VECSMALL` representing a vector over $\mathbf{F}_3$. Specifically `z[0]` is the usual codeword, `z[1]` is the number of components of v and the coefficients are given by pair of adjacent bits of remaining words by increasing indices, with the coding $00 \mapsto 0, 01 \mapsto 1, 10 \mapsto 2$ and 11 is undefined.

`ulong F3v_coeff(GEN x, long i)` returns the coefficient $i \geq 1$ of x.

`void F3v_clear(GEN x, long i)` sets the coefficient $i \geq 1$ of x to 0.

`void F3v_set(GEN x, long i, ulong n)` sets the coefficient $i \geq 1$ of x to $n < 3$,

`ulong F3m_coeff(GEN x, long i, long j)` returns the coefficient (i, j) of x.

`void F3m_set(GEN x, long i, long j, ulong n)` sets the coefficient (i, j) of x to $n < 3$.

`GEN F3m_copy(GEN x)` returns a copy of x.

`GEN F3m_transpose(GEN x)` returns the transpose of x.

`GEN F3m_row(GEN x, long j)` returns the `F3v` which corresponds to the j-th row of the `F3m` x.

`GEN F3m_ker(GEN x)`

`GEN F3m_ker_sp(GEN x, long deplin)`, as `F3m_ker` (if `deplin=0`) or `F2m_deplin` (if `deplin=1`), in place (destroys `x`).

`GEN F3m_mul(GEN x, GEN y)` multiplies `x` and `y` (assumed to have compatible dimensions).

`GEN zero_F3v(long n)` creates a `F3v` with `n` components set to 0.

`GEN zero_F3m_copy(long m, long n)` creates a `F3m` with `m x n` components set to 0.

`GEN F3v_to_Flv(GEN x)`

`GEN ZV_to_F3v(GEN x)`

`GEN RgV_to_F3v(GEN x)`

`GEN F3c_to_ZC(GEN x)`

`GEN F3m_to_Flm(GEN x)`

`GEN F3m_to_ZM(GEN x)`

`GEN Flv_to_F3v(GEN x)`

`GEN Flm_to_F3m(GEN x)`

`GEN ZM_to_F3m(GEN x)`

`GEN RgM_to_F3m(GEN x)`

7.3.5 FlxqV, FlxqC, FlxqM. See FqV, FqC, FqM operations.

GEN FlxqV_dotproduct(GEN x, GEN y, GEN T, ulong p) as FpV_dotproduct.

GEN FlxqV_dotproduct_pre(GEN x, GEN y, GEN T, ulong p, ulong pi) where *pi* is the pseudoinverse of p, or 0 in which case we assume SMALL_ULONG(p).

GEN FlxM_Flx_add_shallow(GEN x, GEN y, ulong p) as RgM_Rg_add_shallow.

GEN FlxqC_Flxq_mul(GEN x, GEN y, GEN T, ulong p)

GEN FlxqM_Flxq_mul(GEN x, GEN y, GEN T, ulong p)

GEN FlxqM_FlxqC_gauss(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_FlxqC_invimage(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_FlxqC_mul(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_deplin(GEN x, GEN T, ulong p)

GEN FlxqM_det(GEN x, GEN T, ulong p)

GEN FlxqM_gauss(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_image(GEN x, GEN T, ulong p)

GEN FlxqM_indexrank(GEN x, GEN T, ulong p)

GEN FlxqM_inv(GEN x, GEN T, ulong p)

GEN FlxqM_invimage(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM_ker(GEN x, GEN T, ulong p)

GEN FlxqM_mul(GEN a, GEN b, GEN T, ulong p)

long FlxqM_rank(GEN x, GEN T, ulong p)

GEN FlxqM_suppl(GEN x, GEN T, ulong p)

GEN matid_FlxqM(long n, GEN T, ulong p)

7.3.6 FpX. Let p an understood t_INT, to be given in the function arguments; in practice p is not assumed to be prime, but be wary. Recall than an Fp object is a t_INT, preferably belonging to $[0, \mathrm{p}-1]$; an FpX is a t_POL in a fixed variable whose coefficients are Fp objects. Unless mentioned otherwise, all outputs in this section are FpXs. All operations are understood to take place in $(\mathbf{Z}/\mathrm{p}\mathbf{Z})[X]$.

7.3.6.1 Conversions. In what follows p is always a t_INT, not necessarily prime.

int RgX_is_FpX(GEN z, GEN *p), z a t_POL, checks if it can be mapped to a FpX, by checking Rg_is_Fp coefficientwise.

GEN RgX_to_FpX(GEN z, GEN p), z a t_POL, returns the FpX obtained by applying Rg_to_Fp coefficientwise.

GEN FpX_red(GEN z, GEN p), z a ZX, returns lift(z * Mod(1,p)), normalized.

GEN FpXV_red(GEN z, GEN p), z a t_VEC of ZX. Applies FpX_red componentwise and returns the result (and we obtain a vector of FpXs).

GEN FpXT_red(GEN z, GEN p), z a tree of ZX. Applies FpX_red to each leaf and returns the result (and we obtain a tree of FpXs).

7.3.6.2 Basic operations. In what follows `p` is always a `t_INT`, not necessarily prime.

Now, except for `p`, the operands and outputs are all `FpX` objects. Results are undefined on other inputs.

`GEN FpX_add(GEN x, GEN y, GEN p)` adds `x` and `y`.

`GEN FpX_neg(GEN x, GEN p)` returns `−x`, the components are between 0 and p if this is the case for the components of x.

`GEN FpX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`GEN FpX_sub(GEN x, GEN y, GEN p)` returns $x - y$.

`GEN FpX_halve(GEN x, GEN p)` returns z such that $2z = x$ modulo p assuming such z exists.

`GEN FpX_mul(GEN x, GEN y, GEN p)` returns xy.

`GEN FpX_mulspec(GEN a, GEN b, GEN p, long na, long nb)` see `ZX_mulspec`

`GEN FpX_sqr(GEN x, GEN p)` returns $\mathtt{x}^2$.

`GEN FpX_powu(GEN x, ulong n, GEN p)` returns x^n.

`GEN FpX_convol(GEN x, GEN y, GEN p)` return the-term by-term product of x and y.

`GEN FpX_divrem(GEN x, GEN y, GEN p, GEN *pr)` returns the quotient of `x` by `y`, and sets `pr` to the remainder.

`GEN FpX_div(GEN x, GEN y, GEN p)` returns the quotient of `x` by `y`.

`GEN FpX_div_by_X_x(GEN A, GEN a, GEN p, GEN *r)` returns the quotient of the `FpX A` by $(X - \mathtt{a})$, and sets `r` to the remainder `A(a)`.

`GEN FpX_rem(GEN x, GEN y, GEN p)` returns the remainder `x` mod `y`.

`long FpX_valrem(GEN x, GEN t, GEN p, GEN *r)` The arguments `x` and `e` being nonzero `FpX` returns the highest exponent e such that $\mathtt{t}^e$ divides `x`. The quotient $\mathtt{x}/\mathtt{t}^e$ is returned in `*r`. In particular, if `t` is irreducible, this returns the valuation at `t` of `x`, and `*r` is the prime-to-`t` part of `x`.

`GEN FpX_deriv(GEN x, GEN p)` returns the derivative of `x`. This function is not memory-clean, but nevertheless suitable for `gerepileupto`.

`GEN FpX_integ(GEN x, GEN p)` returns the primitive of `x` whose constant term is 0.

`GEN FpX_digits(GEN x, GEN B, GEN p)` returns a vector of `FpX` $[c_0, \ldots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^{n} c_i B^i$.

`GEN FpXV_FpX_fromdigits(GEN v, GEN B, GEN p)` where $v = [c_0, \ldots, c_n]$ is a vector of `FpX`, returns $\sum_{i=0}^{n} c_i B^i$.

`GEN FpX_translate(GEN P, GEN c, GEN p)` let c be an `Fp` and let P be an `FpX`; returns the translated `FpX` of $P(X + c)$.

`GEN FpX_gcd(GEN x, GEN y, GEN p)` returns a (not necessarily monic) greatest common divisor of x and y.

`GEN FpX_halfgcd(GEN x, GEN y, GEN p)` returns a two-by-two `FpXM` M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

GEN FpX_extgcd(GEN x, GEN y, GEN p, GEN *u, GEN *v) returns $d = \mathrm{GCD}(\mathtt{x}, \mathtt{y})$ (not necessarily monic), and sets *u, *v to the Bezout coefficients such that $*\mathtt{ux} + *\mathtt{vy} = d$. If *u is set to NULL, it is not computed which is a bit faster. This is useful when computing the inverse of y modulo x.

GEN FpX_center(GEN z, GEN p, GEN pov2) returns the polynomial whose coefficient belong to the symmetric residue system. Assumes the coefficients already belong to $]-p/2, p[$ and that pov2 is shifti(p,-1).

GEN FpX_center_i(GEN z, GEN p, GEN pov2) internal variant of FpX_center, not gerepile-safe.

GEN FpX_Frobenius(GEN T, GEN p) returns $X^p \pmod{T(X)}$.

GEN FpX_matFrobenius(GEN T, GEN p) returns the matrix of the Frobenius automorphism $x \mapsto x^p$ over the power basis of $\mathbf{F}_p[X]/(T)$.

7.3.6.3 Mixed operations. The following functions implement arithmetic operations between FpX and Fp operands, the result being of type FpX. The integer p need not be prime.

GEN Z_to_FpX(GEN x, GEN p, long v) converts a t_INT to a scalar polynomial in variable v, reduced modulo p.

GEN FpX_Fp_add(GEN y, GEN x, GEN p) add the Fp x to the FpX y.

GEN FpX_Fp_add_shallow(GEN y, GEN x, GEN p) add the Fp x to the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN FpX_Fp_sub(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y.

GEN FpX_Fp_sub_shallow(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN Fp_FpX_sub(GEN x, GEN y, GEN p) returns $x - y$, where x is a t_INT and y an FpX.

GEN FpX_Fp_mul(GEN x, GEN y, GEN p) multiplies the FpX x by the Fp y.

GEN FpX_Fp_mulspec(GEN x, GEN y, GEN p, long lx) see ZX_mulspec

GEN FpX_mulu(GEN x, ulong y, GEN p) multiplies the FpX x by y.

GEN FpX_Fp_mul_to_monic(GEN y, GEN x, GEN p) returns yx assuming the result is monic of the same degree as y (in particular $x \neq 0$).

GEN FpX_Fp_div(GEN x, GEN y, GEN p) divides the FpX x by the Fp y.

GEN FpX_divu(GEN x, ulong y, GEN p) divides the FpX x by y.

7.3.6.4 Miscellaneous operations.

GEN FpX_normalize(GEN z, GEN p) divides the FpX z by its leading coefficient. If the latter is 1, z itself is returned, not a copy. If not, the inverse remains uncollected on the stack.

GEN FpX_invBarrett(GEN T, GEN p), returns the Barrett inverse M of T defined by $M(x)x^n \times T(1/x) \equiv 1 \pmod{x^{n-1}}$ where n is the degree of T.

GEN FpX_rescale(GEN P, GEN h, GEN p) returns $h^{\deg(P)}P(x/h)$. P is an FpX and h is a nonzero Fp (the routine would work with any nonzero t_INT but is not efficient in this case). Neither memory-clean nor suitable for gerepileupto.

GEN FpX_eval(GEN x, GEN y, GEN p) evaluates the FpX x at the Fp y. The result is an Fp.

GEN FpX_FpV_multieval(GEN P, GEN v, GEN p) returns the vector $[P(v[1]), \ldots, P(v[n])]$ as a FpV.

GEN FpX_dotproduct(GEN x, GEN y, GEN p) return the scalar product $\sum_{i\geq 0} x_i y_i$ of the coefficients of x and y.

GEN FpXV_FpC_mul(GEN V, GEN W, GEN p) multiplies a nonempty line vector ofFpX by a column vector of Fp of compatible dimensions. The result is an FpX.

GEN FpXV_prod(GEN V, GEN p), V being a vector of FpX, returns their product.

GEN FpXV_factorback(GEN L, GEN e, GEN p, long v) returns $\prod_i L_i^{e_i}$ where L is a vector of FpXs in the variable v and e a vector of non-negative t_INTs or a t_VECSMALL.

GEN FpV_roots_to_pol(GEN V, GEN p, long v), V being a vector of INTs, returns the monic FpX $\prod_i$(pol_x[v] − V[i]).

GEN FpX_chinese_coprime(GEN x, GEN y, GEN Tx, GEN Ty, GEN Tz, GEN p): returns an FpX, congruent to x mod Tx and to y mod Ty. Assumes Tx and Ty are coprime, and Tz = Tx * Ty or NULL (in which case it is computed within).

GEN FpV_polint(GEN x, GEN y, GEN p, long v) returns the FpX interpolation polynomial with value y[i] at x[i]. Assumes lengths are the same, components are t_INTs, and the x[i] are distinct modulo p.

GEN FpV_FpM_polint(GEN x, GEN V, GEN p, long v) equivalent (but faster) to applying FpV_polint(x,...) to all the elements of the vector V (thus, returns a FpXV).

GEN FpX_FpXV_multirem(GEN A, GEN P, GEN p) given a FpX A and a vector P of pairwise coprime FpX of length $n \geq 1$, return a vector B of the same length such that $B[i] = A \pmod{P[i]}$ and $B[i]$ of minimal degree for all $1 \leq i \leq n$.

GEN FpXV_chinese(GEN A, GEN P, GEN p, GEN *pM) let P be a vector of pairwise coprime FpX, let A be a vector of FpX of the same length $n \geq 1$ and let M be the product of the elements of P. Returns a FpX of minimal degree congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If pM is not NULL, set *pM to M.

GEN FpV_invVandermonde(GEN L, GEN d, GEN p) L being a FpV of length n, return the inverse M of the Vandermonde matrix attached to the elements of L, eventually multiplied by d if it is not NULL. If A is a FpV and $B = MA$, then the polynomial $P = \sum_{i=1}^{n} B[i]X^{i-1}$ verifies $P(L[i]) = dA[i]$ for $1 \leq i \leq n$.

int FpX_is_squarefree(GEN f, GEN p) returns 1 if the FpX f is squarefree, 0 otherwise.

int FpX_is_irred(GEN f, GEN p) returns 1 if the FpX f is irreducible, 0 otherwise. Assumes that p is prime. If f has few factors, FpX_nbfact(f,p) == 1 is much faster.

int FpX_is_totally_split(GEN f, GEN p) returns 1 if the FpX f splits into a product of distinct linear factors, 0 otherwise. Assumes that p is prime. The 0 polynomial is not totally split.

long FpX_ispower(GEN f, ulong k, GEN p, GEN *pt) return 1 if the FpX f is a k-th power, 0 otherwise. If pt is not NULL, set it to g such that $g^k = f$.

GEN FpX_factor(GEN f, GEN p), factors the FpX f. Assumes that p is prime. The returned value v is a t_VEC with two components: v[1] is a vector of distinct irreducible (FpX) factors, and v[2] is a t_VECSMALL of corresponding exponents. The order of the factors is deterministic (the computation is not).

GEN FpX_factor_squarefree(GEN f, GEN p) returns the squarefree factorization of f modulo p. This is a vector $[u_1, \ldots, u_k]$ of squarefree and pairwise coprime FpX such that $u_k \neq 1$ and $f = \prod u_i^i$. The other u_i may equal 1. Shallow function.

GEN FpX_ddf(GEN f, GEN p) assuming that f is squarefree, returns the distinct degree factorization of f modulo p. The returned value v is a t_VEC with two components: F=v[1] is a vector of (FpX) factors, and E=v[2] is a t_VECSMALL, such that f is equal to the product of the F[i] and each F[i] is a product of irreducible factors of degree E[i].

long FpX_ddf_degree(GEN f, GEN XP, GEN p) assuming that f is squarefree and that all its factors have the same degree, return the common degree, where XP is FpX_Frobenius(f, p).

long FpX_nbfact(GEN f, GEN p), assuming the FpX f is squarefree, returns the number of its irreducible factors. Assumes that p is prime.

long FpX_nbfact_Frobenius(GEN f, GEN XP, GEN p), as FpX_nbfact(f, p) but faster, where XP is FpX_Frobenius(f, p).

GEN FpX_degfact(GEN f, GEN p), as FpX_factor, but the degrees of the irreducible factors are returned instead of the factors themselves (as a t_VECSMALL). Assumes that p is prime.

long FpX_nbroots(GEN f, GEN p) returns the number of distinct roots in $\mathbf{Z/pZ}$ of the FpX f. Assumes that p is prime.

GEN FpX_oneroot(GEN f, GEN p) returns one root in $\mathbf{Z/pZ}$ of the FpX f. Return NULL if no root exists. Assumes that p is prime.

GEN FpX_oneroot_split(GEN f, GEN p) as FpX_oneroot. Faster when f is close to be totally split.

GEN FpX_roots(GEN f, GEN p) returns the roots in $\mathbf{Z/pZ}$ of the FpX f (without multiplicity, as a vector of Fps). Assumes that p is prime.

GEN FpX_roots_mult(GEN f, long n, GEN p) returns the roots in $\mathbf{Z/pZ}$ with multiplicity at least n of the FpX f (without multiplicity, as a vector of Fps). Assumes that p is prime.

GEN FpX_split_part(GEN f, GEN p) returns the largest totally split squarefree factor of f.

GEN FpX_factcyclo(ulong n, GEN p, ulong m) returns the factors of the n-th cyclotomic polynomial over Fp. if $m = 1$ returns a single factor.

GEN random_FpX(long d, long v, GEN p) returns a random FpX in variable v, of degree less than d.

GEN FpX_resultant(GEN x, GEN y, GEN p) returns the resultant of x and y, both FpX. The result is a t_INT belonging to $[0, p-1]$.

GEN FpX_disc(GEN x, GEN p) returns the discriminant of the FpX x. The result is a t_INT belonging to $[0, p-1]$.

GEN FpX_FpXY_resultant(GEN a, GEN b, GEN p), a a t_POL of t_INTs (say in variable X), b a t_POL (say in variable X) whose coefficients are either t_POLs in $\mathbf{Z}[Y]$ or t_INTs. Returns $\mathrm{Res}_X(a, b)$ in $\mathbf{F}_p[Y]$ as an FpY. The function assumes that X has lower priority than Y.

GEN FpX_Newton(GEN x, long n, GEN p) return $\sum i = 0^{n-1} \pi_i X^i$ where π_i is the sum of the ith-power of the roots of x in an algebraic closure.

GEN FpX_fromNewton(GEN x, GEN p) recover a polynomial from it s Newton sums given by the coefficients of x. This function assumes that p and the accuracy of x as a FpXn is larger than the degree of the solution.

GEN FpX_Laplace(GEN x, GEN p) return $\sum_{i=0}^{n-1} x_i i! X^i$.

GEN FpX_invLaplace(GEN x, GEN p) return $\sum_{i=0}^{n-1} x_i / i! X^i$.

7.3.7 FpXQ, Fq. Let p a t_INT and T an FpX for p, both to be given in the function arguments; an FpXQ object is an FpX whose degree is strictly less than the degree of T. An Fq is either an FpXQ or an Fp. Both represent a class in $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$, in which all operations below take place. In addition, Fq routines also allow T = NULL, in which case no reduction mod T is performed on the result.

For efficiency, the routines in this section may leave small unused objects behind on the stack (their output is still suitable for gerepileupto). Besides T and p, arguments are either FpXQ or Fq depending on the function name. (All Fq routines accept FpXQs by definition, not the other way round.)

7.3.7.1 Preconditioned reduction.

For faster reduction, the modulus T can be replaced by an extended modulus in all FpXQ- and Fq-classes functions, and in FpX_rem and FpX_divrem. An extended modulus(FpXT, which is a tree whose leaves are FpX)In current implementation, an extended modulus is either a plain modulus (an FpX) or a pair of polynomials, one being the plain modulus T and the other being FpX_invBarret(T, p).

GEN FpX_get_red(GEN T, GEN p) returns the extended modulus eT.

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get_FpX_mod(GEN eT) returns the underlying modulus T.

GEN get_FpX_var(GEN eT) returns the variable number varn(T).

GEN get_FpX_degree(GEN eT) returns the degree degpol(T).

7.3.7.2 Conversions.

int ff_parse_Tp(GEN Tp, GEN *T, GEN *p, long red) Tp is either a prime number p or a t_VEC with 2 entries T (an irreducible polynomial mod p) and p (a prime number). Sets *p and *T to the corresponding GENs (NULL if undefined). If red is nonzero, normalize *T as an FpX; on the other hand, to initialize a p-adic function, set red to 0 and *T is left as is and must be a ZX to start with. Return 1 on success, and 0 on failure. This helper routine is used by GP functions such as factormod where a single user argument defines a finite field. t_FFELT is not supported.

GEN Rg_is_FpXQ(GEN z, GEN *T, GEN *p), checks if z is a GEN which can be mapped to $\mathbf{F}_p[X]/(T)$: anything for which Rg_is_Fp return 1, a t_POL for which RgX_to_FpX return 1, a t_POLMOD whose modulus is equal to *T if *T is not NULL (once mapped to a FpX), or a t_FFELT z with the same definition field as *T if *T is not NULL and is a t_FFELT.

If an integer modulus is found it is put in *p, else *p is left unchanged. If a polynomial modulus is found it is put in *T, if a t_FFELT z is found, z is put in *T, else *T is left unchanged.

int RgX_is_FpXQX(GEN z, GEN *T, GEN *p), z a t_POL, checks if it can be mapped to a FpXQX, by checking Rg_is_FpXQ coefficientwise.

GEN Rg_to_FpXQ(GEN z, GEN T, GEN p), z a GEN which can be mapped to $\mathbf{F}_p[X]/(T)$: anything Rg_to_Fp can be applied to, a t_POL to which RgX_to_FpX can be applied to, a t_POLMOD whose modulus is divisible by T (once mapped to a FpX), a suitable t_RFRAC. Returns z as an FpXQ, normalized.

GEN Rg_to_Fq(GEN z, GEN T, GEN p), applies Rg_to_Fp if T is NULL and Rg_to_FpXQ otherwise.

GEN RgX_to_FpXQX(GEN z, GEN T, GEN p), z a t_POL, returns the FpXQ obtained by applying Rg_to_FpXQ coefficientwise.

GEN RgX_to_FqX(GEN z, GEN T, GEN p): let z be a t_POL; returns the FqX obtained by applying Rg_to_Fq coefficientwise.

GEN Fq_to_FpXQ(GEN z, GEN T, GEN p /*unused*/) if z is a t_INT, convert it to a constant polynomial in the variable of T, otherwise return z (shallow function).

GEN Fq_red(GEN x, GEN T, GEN p), x a ZX or t_INT, reduce it to an Fq (T = NULL is allowed iff x is a t_INT).

GEN FqX_red(GEN x, GEN T, GEN p), x a t_POL whose coefficients are ZXs or t_INTs, reduce them to Fqs. (If T = NULL, as FpXX_red(x, p).)

GEN FqV_red(GEN x, GEN T, GEN p), x a vector of ZXs or t_INTs, reduce them to Fqs. (If T = NULL, only reduce components mod p to FpXs or Fps.)

GEN FpXQ_red(GEN x, GEN T, GEN p) x a t_POL whose coefficients are t_INTs, reduce them to FpXQs.

7.3.8 FpXQ.

GEN FpXQ_add(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_sub(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_mul(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_sqr(GEN x, GEN T, GEN p)

GEN FpXQ_div(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_inv(GEN x, GEN T, GEN p) computes the inverse of x

GEN FpXQ_invsafe(GEN x, GEN T, GEN p), as FpXQ_inv, returning NULL if x is not invertible.

GEN FpXQ_pow(GEN x, GEN n, GEN T, GEN p) computes $\mathtt{x}^{\mathtt{n}}$.

GEN FpXQ_powu(GEN x, ulong n, GEN T, GEN p) computes $\mathtt{x}^{\mathtt{n}}$ for small n.

In the following three functions the integer parameter ord can be given either as a positive t_INT N, or as its factorization matrix faN, or as a pair $[N, faN]$. The parameter may be omitted by setting it to NULL (the value is then $p^d - 1$, $d = \deg T$).

GEN FpXQ_log(GEN a, GEN g, GEN ord, GEN T, GEN p) Let g be of order dividing ord in the finite field $\mathbf{F}_p[X]/(T)$, return e such that $a^e = g$. If e does not exists, the result is undefined. Assumes that T is irreducible mod p.

GEN Fp_FpXQ_log(GEN a, GEN g, GEN ord, GEN T, GEN p) As FpXQ_log, a being a Fp.

GEN FpXQ_order(GEN a, GEN ord, GEN T, GEN p) returns the order of the FpXQ a. Assume that ord is a multiple of the order of a. Assume that T is irreducible mod p.

`int FpXQ_issquare(GEN x, GEN T, GEN p)` returns 1 if x is a square and 0 otherwise. Assumes that `T` is irreducible mod `p`.

`GEN FpXQ_sqrt(GEN x, GEN T, GEN p)` returns a square root of `x`. Return `NULL` if `x` is not a square.

`GEN FpXQ_sqrtn(GEN x, GEN n, GEN T, GEN p, GEN *zn)` Let Tbe irreducible mod p and $q = p^{\deg T}$; returns `NULL` if a is not an n-th power residue mod p. Otherwise, returns an n-th root of a; if `zn` is not `NULL` set it to a primitive m-th root of 1 in $\mathbf{F}_q$, $m = \gcd(q-1, n)$ allowing to compute all m solutions in $\mathbf{F}_q$ of the equation $x^n = a$.

7.3.9 Fq.

`GEN Fq_add(GEN x, GEN y, GEN T/*unused*/, GEN p)`

`GEN Fq_sub(GEN x, GEN y, GEN T/*unused*/, GEN p)`

`GEN Fq_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN Fq_Fp_mul(GEN x, GEN y, GEN T, GEN p)` multiplies the `Fq` x by the `t_INT` y.

`GEN Fq_mulu(GEN x, ulong y, GEN T, GEN p)` multiplies the `Fq` x by the scalar y.

`GEN Fq_halve(GEN x, GEN T, GEN p)` returns z such that $2z = x$ assuming such z exists.

`GEN Fq_sqr(GEN x, GEN T, GEN p)`

`GEN Fq_neg(GEN x, GEN T, GEN p)`

`GEN Fq_neg_inv(GEN x, GEN T, GEN p)` computes $-\mathtt{x}^{-1}$

`GEN Fq_inv(GEN x, GEN pol, GEN p)` computes $\mathtt{x}^{-1}$, raising an error if `x` is not invertible.

`GEN Fq_invsafe(GEN x, GEN pol, GEN p)` as `Fq_inv`, but returns `NULL` if `x` is not invertible.

`GEN Fq_div(GEN x, GEN y, GEN T, GEN p)`

`GEN FqV_inv(GEN x, GEN T, GEN p)` x being a vector of `Fq`s, return the vector of inverses of the $x[i]$. The routine uses Montgomery's trick, and involves a single inversion, plus $3(N-1)$ multiplications for N entries. The routine is not stack-clean: $2N$ `FpXQ` are left on stack, besides the N in the result.

`GEN FqV_factorback(GEN L, GEN e, GEN T, GEN p)` given an `FqV` L and a `ZV` or `zv` e of the same length, return $\prod_i L_i^{e_i}$ modulo p.

`GEN Fq_pow(GEN x, GEN n, GEN pol, GEN p)` returns $\mathtt{x}^{\mathtt{n}}$.

`GEN Fq_powu(GEN x, ulong n, GEN pol, GEN p)` returns $\mathtt{x}^{\mathtt{n}}$ for small n.

`GEN Fq_log(GEN a, GEN g, GEN ord, GEN T, GEN p)` as `Fp_log` or `FpXQ_log`.

`int Fq_issquare(GEN x, GEN T, GEN p)` returns 1 if x is a square and 0 otherwise. Assumes that `T` is irreducible mod `p` and that p is prime; $T =$ `NULL` is forbidden unless x is an `Fp`.

`long Fq_ispower(GEN x, GEN n, GEN T, GEN p)` returns 1 if x is a n-th power and 0 otherwise. Assumes that `T` is irreducible mod `p` and that p is prime; $T =$ `NULL` is forbidden unless x is an `Fp`.

`GEN Fq_sqrt(GEN x, GEN T, GEN p)` returns a square root of `x`. Return `NULL` if `x` is not a square.

`GEN Fq_sqrtn(GEN a, GEN n, GEN T, GEN p, GEN *zn)` as `FpXQ_sqrtn`.

GEN FpXQ_charpoly(GEN x, GEN T, GEN p) returns the characteristic polynomial of x

GEN FpXQ_minpoly(GEN x, GEN T, GEN p) returns the minimal polynomial of x

GEN FpXQ_norm(GEN x, GEN T, GEN p) returns the norm of x

GEN FpXQ_trace(GEN x, GEN T, GEN p) returns the trace of x

GEN FpXQ_conjvec(GEN x, GEN T, GEN p) returns the vector of conjugates $[x, x^p, x^{p^2}, \ldots, x^{p^{n-1}}]$ where n is the degree of T.

GEN gener_FpXQ(GEN T, GEN p, GEN *po) returns a primitive root modulo (T, p). T is an FpX assumed to be irreducible modulo the prime p. If po is not NULL it is set to $[o, fa]$, where o is the order of the multiplicative group of the finite field, and fa is its factorization.

GEN gener_FpXQ_local(GEN T, GEN p, GEN L), L being a vector of primes dividing $p^{\deg T} - 1$, returns an element of $G := \mathbf{F}_p[x]/(T)$ which is a generator of the ℓ-Sylow of G for every ℓ in L. It is not necessary, and in fact slightly inefficient, to include $\ell = 2$, since 2 is treated separately in any case, i.e. the generator obtained is never a square if p is odd.

GEN gener_Fq_local(GEN T, GEN p, GEN L) as pgener_Fp_local(p, L) if T is NULL, or gener_FpXQ_local (otherwise).

GEN FpXQ_powers(GEN x, long n, GEN T, GEN p) returns $[\mathtt{x}^0, \ldots, \mathtt{x}^\mathtt{n}]$ as a t_VEC of FpXQs.

GEN FpXQ_matrix_pow(GEN x, long m, long n, GEN T, GEN p), as FpXQ_powers$(x, n-1, T, p)$, but returns the powers as a an $m \times n$ matrix. Usually, we have $m = n = \deg T$.

GEN FpXQ_autpow(GEN a, ulong n, GEN T, GEN p) computes $\sigma^n(X)$ assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_p[X]/T(X)$.

GEN FpXQ_autsum(GEN a, ulong n, GEN T, GEN p) a being a two-component vector, σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b\sigma(b) \ldots \sigma^{n-1}(b)]$ where $b = a[2]$.

GEN FpXQ_auttrace(GEN a, ulong n, GEN T, GEN p) a being a two-component vector, σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b + \sigma(b) + \ldots + \sigma^{n-1}(b)]$ where $b = a[2]$.

GEN FpXQ_autpowers(GEN S, long n, GEN T, GEN p) returns $[x, S(x), S(S(x)), \ldots, S^{(n)}(x)]$ as a t_VEC of FpXQs.

GEN FpXQM_autsum(GEN a, long n, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b\sigma(b) \ldots \sigma^{n-1}(b)]$ where $b = a[2]$ is a square matrix.

GEN FpX_FpXQ_eval(GEN f, GEN x, GEN T, GEN p) returns f(x).

GEN FpX_FpXQV_eval(GEN f, GEN V, GEN T, GEN p) returns f(x), assuming that V was computed by FpXQ_powers$(\mathtt{x}, n, \mathtt{T}, \mathtt{p})$.

GEN FpXC_FpXQ_eval(GEN C, GEN x, GEN T, GEN p) applies FpX_FpXQV_eval to all elements of the vector C and returns a t_COL.

GEN FpXC_FpXQV_eval(GEN C, GEN V, GEN T, GEN p) applies FpX_FpXQV_eval to all elements of the vector C and returns a t_COL.

GEN FpXM_FpXQV_eval(GEN M, GEN V, GEN T, GEN p) applies FpX_FpXQV_eval to all elements of the matrix M.

7.3.10 `FpXn`. Let `p` a `t_INT` and `T` an `FpX` for `p`, both to be given in the function arguments; an `FpXn` object is an `FpX` whose degree is strictly less than `n`. They represent a class in $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$, in which all operations below take place. They can be seen as truncated power series.

`GEN FpXn_mul(GEN x, GEN y, long n, GEN p)` return $xy \pmod{X^n}$.

`GEN FpXn_sqr(GEN x, long n, GEN p)` return $x^2 \pmod{X^n}$.

`GEN FpXn_div(GEN x, GEN y, long n, GEN p)` return $x/y \pmod{X^n}$.

`GEN FpXn_inv(GEN x, long n, GEN p)` return $1/x \pmod{X^n}$.

`GEN FpXn_exp(GEN f, long n, GEN p)` return $\exp(f)$ as a composition of formal power series. It is required that the valuation of f is positive and that $p > n$.

`GEN FpXn_expint(GEN f, long n, GEN p)` return $\exp(F)$ where F is the primitive of f that vanishes at 0. It is required that $p > n$.

7.3.11 `FpXC`, `FpXM`.

`GEN FpXC_center(GEN C, GEN p, GEN pov2)`

`GEN FpXM_center(GEN M, GEN p, GEN pov2)`

7.3.12 `FpXX`, `FpXY`. Contrary to what the name implies, an `FpXX` is a `t_POL` whose coefficients are either `t_INT`s or `FpX`s. This reduces memory overhead at the expense of consistency. The prefix `FpXY` is an alias for `FpXX` when variables matters.

`GEN FpXX_red(GEN z, GEN p)`, `z` a `t_POL` whose coefficients are either `ZX`s or `t_INT`s. Returns the `t_POL` equal to `z` with all components reduced modulo `p`.

`GEN FpXX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`GEN FpXX_add(GEN x, GEN y, GEN p)` adds `x` and `y`.

`GEN FpXX_sub(GEN x, GEN y, GEN p)` returns `x` − `y`.

`GEN FpXX_neg(GEN x, GEN p)` returns −`x`.

`GEN FpXX_Fp_mul(GEN x, GEN y, GEN p)` multiplies the `FpXX` `x` by the `Fp` `y`.

`GEN FpXX_FpX_mul(GEN x, GEN y, GEN p)` multiplies the coefficients of the `FpXX` `x` by the `FpX` `y`.

`GEN FpXX_mulu(GEN x, GEN y, GEN p)` multiplies the `FpXX` `x` by the scalar `y`.

`GEN FpXX_halve(GEN x, GEN p)` returns z such that $2z = x$ assuming such z exists.

`GEN FpXX_deriv(GEN P, GEN p)` differentiates `P` with respect to the main variable.

`GEN FpXX_integ(GEN P, GEN p)` returns the primitive of `P` with respect to the main variable whose constant term is 0.

`GEN FpXY_eval(GEN Q, GEN y, GEN x, GEN p)` Q being an `FpXY`, i.e. a `t_POL` with `Fp` or `FpX` coefficients representing an element of $\mathbf{F}_p[X][Y]$. Returns the `Fp` $Q(x,y)$.

`GEN FpXY_evalx(GEN Q, GEN x, GEN p)` Q being an `FpXY`, returns the `FpX` $Q(x,Y)$, where Y is the main variable of Q.

`GEN FpXY_evaly(GEN Q, GEN y, GEN p, long vx)` Q an `FpXY`, returns the `FpX` $Q(X,y)$, where X is the second variable of Q, and `vx` is the variable number of X.

GEN FpXY_FpXQ_evaly(GEN Q, GEN y, GEN T, GEN p, long vx) Q an FpXY and y being an FpXQ, returns the FpXQX $Q(X,y)$, where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY_Fq_evaly(GEN Q, GEN y, GEN T, GEN p, long vx) Q an FpXY and y being an Fq, returns the FqX $Q(X,y)$, where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY_FpXQ_evalx(GEN Q, GEN x, ulong p) Q an FpXY and x being an FpXQ, returns the FpXQX $Q(x,Y)$, where Y is the first variable of Q.

GEN FpXY_FpXQV_evalx(GEN Q, GEN V, ulong p) Q an FpXY and x being an FpXQ, returns the FpXQX $Q(x,Y)$, where Y is the first variable of Q, assuming that V was computed by FpXQ_powers(x, n, T, p).

GEN FpXYQQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x being a FpXY, T being a FpX and S being a FpY, return $x^n \pmod{S, T, p}$.

7.3.13 FpXQX, FqX. Contrary to what the name implies, an FpXQX is a t_POL whose coefficients are Fqs. So the only difference between FqX and FpXQX routines is that T = NULL is not allowed in the latter. (It was thought more useful to allow t_INT components than to enforce strict consistency, which would not imply any efficiency gain.)

7.3.13.1 Basic operations.

GEN FqX_add(GEN x, GEN y, GEN T, GEN p)

GEN FqX_Fq_add(GEN x, GEN y, GEN T, GEN p) adds the Fq y to the FqX x.

GEN FqX_Fq_sub(GEN x, GEN y, GEN T, GEN p) substracts the Fq y to the FqX x.

GEN FqX_neg(GEN x, GEN T, GEN p)

GEN FqX_sub(GEN x, GEN y, GEN T, GEN p)

GEN FqX_mul(GEN x, GEN y, GEN T, GEN p)

GEN FqX_Fq_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX x by the Fq y.

GEN FqX_mulu(GEN x, ulong y, GEN T, GEN p) multiplies the FqX x by the scalar y.

GEN FqX_halve(GEN x, GEN T, GEN p) returns z such that $2z = x$ assuming such z exists.

GEN FqX_Fp_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX x by the t_INT y.

GEN FqX_Fq_mul_to_monic(GEN x, GEN y, GEN T, GEN p) returns xy assuming the result is monic of the same degree as x (in particular $y \neq 0$).

GEN FpXQX_normalize(GEN z, GEN T, GEN p)

GEN FqX_normalize(GEN z, GEN T, GEN p) divides the FqX z by its leading term. The leading coefficient becomes 1 as a t_INT.

GEN FqX_sqr(GEN x, GEN T, GEN p)

GEN FqX_powu(GEN x, ulong n, GEN T, GEN p)

GEN FqX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *z)

GEN FqX_div(GEN x, GEN y, GEN T, GEN p)

GEN FqX_div_by_X_x(GEN a, GEN x, GEN T, GEN p, GEN *r)

`GEN FqX_rem(GEN x, GEN y, GEN T, GEN p)`

`GEN FqX_deriv(GEN x, GEN T, GEN p)` returns the derivative of `x`. (This function is suitable for `gerepilupto` but not memory-clean.)

`GEN FqX_integ(GEN x, GEN T, GEN p)` returns the primitive of `x`. whose constant term is 0.

`GEN FqX_translate(GEN P, GEN c, GEN T, GEN p)` let c be an `Fq` defined modulo (p, T), and let P be an `FqX`; returns the translated `FqX` of $P(X + c)$.

`GEN FqX_gcd(GEN P, GEN Q, GEN T, GEN p)` returns a (not necessarily monic) greatest common divisor of x and y.

`GEN FqX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)` returns $d = \mathrm{GCD}(\mathtt{x}, \mathtt{y})$ (not necessarily monic), and sets `*u`, `*v` to the Bezout coefficients such that $*\mathtt{ux} + *\mathtt{vy} = d$.

`GEN FqX_halfgcd(GEN x, GEN y, GEN T, GEN p)` returns a two-by-two `FqXM` M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

`GEN FqX_eval(GEN x, GEN y, GEN T, GEN p)` evaluates the `FqX` `x` at the `Fq` `y`. The result is an `Fq`.

`GEN FqXY_eval(GEN Q, GEN y, GEN x, GEN T, GEN p)` Q an `FqXY`, i.e. a `t_POL` with `Fq` or `FqX` coefficients representing an element of $\mathbf{F}_q[X][Y]$. Returns the `Fq` $Q(x, y)$.

`GEN FqXY_evalx(GEN Q, GEN x, GEN T, GEN p)` Q being an `FqXY`, returns the `FqX` $Q(x, Y)$, where Y is the main variable of Q.

`GEN random_FpXQX(long d, long v, GEN T, GEN p)` returns a random `FpXQX` in variable `v`, of degree less than `d`.

`GEN FpXQX_renormalize(GEN x, long lx)`

`GEN FpXQX_red(GEN z, GEN T, GEN p)` `z` a `t_POL` whose coefficients are `ZX`s or `t_INT`s, reduce them to `FpXQ`s.

`GEN FpXQXV_red(GEN z, GEN T, GEN p)`, `z` a `t_VEC` of `ZXX`. Applies `FpX_red` componentwise and returns the result (and we obtain a vector of `FpXQX`s).

`GEN FpXQXT_red(GEN z, GEN T, GEN p)`, `z` a tree of `ZXX`. Applies `FpX_red` to each leaf and returns the result (and we obtain a tree of `FpXQX`s).

`GEN FpXQX_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN Kronecker_to_FpXQX(GEN z, GEN T, GEN p)`. Let $n = \deg T$ and let $P(X, Y) \in \mathbf{Z}[X, Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{F}_p[X]/(T)$ and $\deg_X P < 2n - 1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P (see `RgXX_to_Kronecker`), this function returns $Q \in \mathbf{Z}[X, t]$ such that Q is congruent to $P(X, t) \bmod (p, T(X))$, $\deg_X Q < n$, and all coefficients are in $[0, p[$. Not stack-clean. Note that t need not be the same variable as Y!

`GEN FpXQX_FpXQ_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_sqr(GEN x, GEN T, GEN p)`

`GEN FpXQX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *pr)`

`GEN FpXQX_div(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_div_by_X_x(GEN a, GEN x, GEN T, GEN p, GEN *r)`

GEN FpXQX_rem(GEN x, GEN y, GEN T, GEN p)

GEN FpXQX_powu(GEN x, ulong n, GEN T, GEN p) returns x^n.

GEN FpXQX_digits(GEN x, GEN B, GEN T, GEN p)

GEN FpXQX_dotproduct(GEN x, GEN y, GEN T, GEN p) returns the scalar product of the coefficients of x and y.

GEN FpXQXV_FpXQX_fromdigits(GEN v, GEN B, GEN T, GEN p)

GEN FpXQX_invBarrett(GEN y, GEN T, GEN p) returns the Barrett inverse of the FpXQX y, namely a lift of $1/\texttt{polrecip}(y) + O(x^{\deg(y)-1})$.

GEN FpXQXV_prod(GEN V, GEN T, GEN p), V being a vector of FpXQX, returns their product.

GEN FpXQX_gcd(GEN x, GEN y, GEN T, GEN p)

GEN FpXQX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)

GEN FpXQX_halfgcd(GEN x, GEN y, GEN T, GEN p)

GEN FpXQX_resultant(GEN x, GEN y, GEN T, GEN p) returns the resultant of x and y.

GEN FpXQX_disc(GEN x, GEN T, GEN p) returns the discriminant of x.

GEN FpXQX_FpXQXQ_eval(GEN f, GEN x, GEN S, GEN T, GEN p) returns f(x).

7.3.14 FpXQXn, FqXn.

A FpXQXn is a t_FpXQX which represents an element of the ring $(Fp[X]/T(X))[Y]/(Y^n)$, where T is a FpX.

GEN FpXQXn_sqr(GEN x, long n, GEN T, GEN p)

GEN FqXn_sqr(GEN x, long n, GEN T, GEN p)

GEN FpXQXn_mul(GEN x, GEN y, long n, GEN T, GEN p)

GEN FqXn_mul(GEN x, GEN y, long n, GEN T, GEN p)

GEN FpXQXn_div(GEN x, GEN y, long n, GEN T, GEN p)

GEN FpXQXn_inv(GEN x, long n, GEN T, GEN p)

GEN FqXn_inv(GEN x, long n, GEN T, GEN p)

GEN FpXQXn_exp(GEN x, long n, GEN T, GEN p) return $\exp(x)$ as a composition of formal power series. It is required that the valuation of x is positive and that $p > n$.

GEN FqXn_exp(GEN x, long n, GEN T, GEN p)

GEN FpXQXn_expint(GEN f, long n, GEN p) return $\exp(F)$ where F is the primitive of f that vanishes at 0. It is required that $p > n$.

GEN FqXn_expint(GEN x, long n, GEN T, GEN p)

7.3.15 `FpXQXQ`, `FqXQ`.

A `FpXQXQ` is a `t_FpXQX` which represents an element of the ring $(Fp[X]/T(X))[Y]/S(X,Y)$, where T is a `FpX` and S a `FpXQX` modulo T. A `FqXQ` is identical except that T is allowed to be `NULL` in which case S must be a `FpX`.

7.3.15.1 Preconditioned reduction.

For faster reduction, the modulus `S` can be replaced by an extended modulus, which is an `FpXQXT`, in all `FpXQXQ`- and `FqXQ`-classes functions, and in `FpXQX_rem` and `FpXQX_divrem`.

`GEN FpXQX_get_red(GEN S, GEN T, GEN p)` returns the extended modulus `eS`.

`GEN FqX_get_red(GEN S, GEN T, GEN p)` identical, but allow T to be `NULL`, in which case it returns `FpX_get_red(S,p)`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_FpXQX_mod(GEN eS)` returns the underlying modulus `S`.

`GEN get_FpXQX_var(GEN eS)` returns the variable number of the modulus.

`GEN get_FpXQX_degree(GEN eS)` returns the degree of the modulus.

Furthermore, `ZXXT_to_FlxXT` allows to convert an extended modulus for a `FpXQX` to an extended modulus for the corresponding `FlxqX`.

7.3.15.2 basic operations.

`GEN FpXQX_FpXQXQV_eval(GEN f, GEN V, GEN S, GEN T, GEN p)` returns `f(x)`, assuming that `V` was computed by `FpXQXQ_powers(x,` n`, S, T, p)`.

`GEN FpXQXQ_div(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FpXQX`s, returns $\mathtt{x} * \mathtt{y}^{-1}$ modulo `S`.

`GEN FpXQXQ_inv(GEN x, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $\mathtt{x}^{-1}$ modulo `S`.

`GEN FpXQXQ_invsafe(GEN x, GEN S, GEN T, GEN p)`, as `FpXQXQ_inv`, returning `NULL` if `x` is not invertible.

`GEN FpXQXQ_mul(GEN x, GEN y, GEN S, GEN T, GEN p)`, `x`, `y` and `S` being `FpXQX`s, returns `xy` modulo `S`.

`GEN FpXQXQ_sqr(GEN x, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $\mathtt{x}^2$ modulo `S`.

`GEN FpXQXQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $\mathtt{x}^\mathtt{n}$ modulo `S`.

`GEN FpXQXQ_powers(GEN x, long n, GEN S, GEN T, GEN p)`, `x` and `S` being `FpXQX`s, returns $[\mathtt{x}^0, \ldots, \mathtt{x}^\mathtt{n}]$ as a `t_VEC` of `FpXQXQ`s.

`GEN FpXQXQ_halfFrobenius(GEN A, GEN S, GEN T, GEN p)` returns $A(X)^{(q-1)/2} \pmod{S(X)}$ over the finite field $\mathbf{F}_q$ defined by T and p, thus $q = p^n$ where n is the degree of T.

`GEN FpXQXQ_minpoly(GEN x, GEN S, GEN T, GEN p)`, as `FpXQ_minpoly`

`GEN FpXQXQ_matrix_pow(GEN x, long m, long n, GEN S, GEN T, GEN p)` returns the same powers of `x` as `FpXQXQ_powers`$(x, n-1, S, T, p)$, but as an $m \times n$ matrix.

GEN FpXQXQ_autpow(GEN a, long n, GEN S, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$, returns $[\sigma^n(X), \sigma^n(Y)]$.

GEN FpXQXQ_autsum(GEN a, long n, GEN S, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$, returns the vector $[\sigma^n(X), \sigma^n(Y), b\sigma(b)\ldots\sigma^{n-1}(b)]$ where $b = a[3]$.

GEN FpXQXQ_auttrace(GEN a, long n, GEN S, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = X \pmod{T(X)}$, $\sigma(Y) = a[1] \pmod{S(X,Y), T(X)}$, returns the vector $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \ldots + \sigma^{n-1}(b)]$ where $b = a[2]$.

GEN FqXQ_add(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns x + y modulo S.

GEN FqXQ_sub(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns x − y modulo S.

GEN FqXQ_mul(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns xy modulo S.

GEN FqXQ_div(GEN x, GEN y, GEN S, GEN T, GEN p), x and S being FqXs, returns x/y modulo S.

GEN FqXQ_inv(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns $\mathtt{x}^{-1}$ modulo S.

GEN FqXQ_invsafe(GEN x, GEN S, GEN T, GEN p) , as FqXQ_inv, returning NULL if x is not invertible.

GEN FqXQ_sqr(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns $\mathtt{x}^2$ modulo S.

GEN FqXQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x and S being FqXs, returns $\mathtt{x}^\mathtt{n}$ modulo S.

GEN FqXQ_powers(GEN x, long n, GEN S, GEN T, GEN p), x and S being FqXs, returns $[\mathtt{x}^0, \ldots, \mathtt{x}^\mathtt{n}]$ as a t_VEC of FqXQs.

GEN FqXQ_matrix_pow(GEN x, long m, long n, GEN S, GEN T, GEN p) returns the same powers of x as FqXQ_powers$(x, n-1, S, T, p)$, but as an $m \times n$ matrix.

GEN FqV_roots_to_pol(GEN V, GEN T, GEN p, long v), V being a vector of Fqs, returns the monic FqX $\prod_i$(pol_x[v] − V[i]).

7.3.15.3 Miscellaneous operations.

GEN init_Fq(GEN p, long n, long v) returns an irreducible polynomial of degree $\mathtt{n} > 0$ over $\mathbf{F}_p$, in variable v.

int FqX_is_squarefree(GEN P, GEN T, GEN p)

GEN FpXQX_roots(GEN f, GEN T, GEN p) return the roots of f in $\mathbf{F}_p[X]/(T)$. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FqX_roots(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX_factor(GEN f, GEN T, GEN p) same output convention as FpX_factor. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FqX_factor(GEN f, GEN T, GEN p) same but allow T = NULL.

`GEN FpXQX_factor_squarefree(GEN f, GEN T, GEN p)` squarefree factorization of f modulo (T, p); same output convention as `FpX_factor_squarefree`. Assumes `p` is prime and `T` irreducible in $\mathbf{F}_p[X]$.

`GEN FqX_factor_squarefree(GEN f, GEN T, GEN p)` same but allow `T` = `NULL`.

`GEN FpXQX_ddf(GEN f, GEN T, GEN p)` as `FpX_ddf`.

`GEN FqX_ddf(GEN f, GEN T, GEN p)` same but allow `T` = `NULL`.

`long FpXQX_ddf_degree(GEN f, GEN XP, GEN T, GEN p)`, as `FpX_ddf_degree`.

`GEN FpXQX_degfact(GEN f, GEN T, GEN p)`, as `FpX_degfact`.

`GEN FqX_degfact(GEN f, GEN T, GEN p)` same but allow `T` = `NULL`.

`GEN FpXQX_split_part(GEN f, GEN T, GEN p)` returns the largest totally split squarefree factor of f.

`long FpXQX_ispower(GEN f, ulong k, GEN T, GEN p, GEN *pt)` return 1 if the `FpXQX` f is a k-th power, 0 otherwise. If `pt` is not `NULL`, set it to g such that $g^k = f$.

`long FqX_ispower(GEN f, ulong k, GEN T, GEN p, GEN *pt)` same but allow `T` = `NULL`.

`GEN FpX_factorff(GEN P, GEN T, GEN p)`. Assumes `p` prime and `T` irreducible in $\mathbf{F}_p[X]$. Factor the `FpX` `P` over the finite field $\mathbf{F}_p[Y]/(T(Y))$. See `FpX_factorff_irred` if `P` is known to be irreducible of $\mathbf{F}_p$.

`GEN FpX_rootsff(GEN P, GEN T, GEN p)`. Assumes `p` prime and `T` irreducible in $\mathbf{F}_p[X]$. Returns the roots of the `FpX` `P` belonging to the finite field $\mathbf{F}_p[Y]/(T(Y))$.

`GEN FpX_factorff_irred(GEN P, GEN T, GEN p)`. Assumes `p` prime and `T` irreducible in $\mathbf{F}_p[X]$. Factors the *irreducible* `FpX` `P` over the finite field $\mathbf{F}_p[Y]/(T(Y))$ and returns the vector of irreducible `FqX`s factors (the exponents, being all equal to 1, are not included).

`GEN FpX_ffisom(GEN P, GEN Q, GEN p)`. Assumes `p` prime, `P`, `Q` are `ZX`s, both irreducible mod `p`, and $\deg(P) \mid \deg Q$. Outputs a monomorphism between $\mathbf{F}_p[X]/(P)$ and $\mathbf{F}_p[X]/(Q)$, as a polynomial R such that `Q` $\mid$ `P`(R) in $\mathbf{F}_p[X]$. If `P` and `Q` have the same degree, it is of course an isomorphism.

`void FpX_ffintersect(GEN P, GEN Q, long n, GEN p, GEN *SP, GEN *SQ, GEN MA, GEN MB)` Assumes `p` is prime, `P`, `Q` are `ZX`s, both irreducible mod `p`, and `n` divides both the degree of `P` and `Q`. Compute `SP` and `SQ` such that the subfield of $\mathbf{F}_p[X]/(P)$ generated by `SP` and the subfield of $\mathbf{F}_p[X]/(Q)$ generated by `SQ` are isomorphic of degree `n`. The polynomials `P` and `Q` do not need to be of the same variable. If `MA` (resp. `MB`) is not `NULL`, it must be the matrix of the Frobenius map in $\mathbf{F}_p[X]/(P)$ (resp. $\mathbf{F}_p[X]/(Q)$).

`GEN FpXQ_ffisom_inv(GEN S, GEN T, GEN p)`. Assumes `p` is prime, `T` a `ZX`, which is irreducible modulo `p`, `S` a `ZX` representing an automorphism of $\mathbf{F}_q := \mathbf{F}_p[X]/(\mathtt{T})$. (`S`$(X)$ is the image of X by the automorphism.) Returns the inverse automorphism of `S`, in the same format, i.e. an `FpX` H such that $H(\mathtt{S}) \equiv X$ modulo $(\mathtt{T}, \mathtt{p})$.

`long FpXQX_nbfact(GEN S, GEN T, GEN p)` returns the number of irreducible factors of the polynomial S over the finite field $\mathbf{F}_q$ defined by T and p.

`long FpXQX_nbfact_Frobenius(GEN S, GEN Xq, GEN T, GEN p)` as `FpXQX_nbfact` where `Xq` is `FpXQX_Frobenius(S, T, p)`.

`long FqX_nbfact(GEN S, GEN T, GEN p)` as above but accept `T=NULL`.

`long FpXQX_nbroots(GEN S, GEN T, GEN p)` returns the number of roots of the polynomial S over the finite field $\mathbf{F}_q$ defined by T and p.

`long FqX_nbroots(GEN S, GEN T, GEN p)` as above but accept `T=NULL`.

`GEN FpXQX_Frobenius(GEN S, GEN T, GEN p)` returns $X^q \pmod{S(X)}$ over the finite field $\mathbf{F}_q$ defined by T and p, thus $q = p^n$ where n is the degree of T.

7.3.16 `Flx`. Let `p` be an `ulong`, not assumed to be prime unless mentioned otherwise (e.g., all functions involving Euclidean divisions and factorizations), to be given the function arguments; an `Fl` is an `ulong` belonging to $[0, \mathtt{p}-1]$, an `Flx z` is a `t_VECSMALL` representing a polynomial with small integer coefficients. Specifically `z[0]` is the usual codeword, `z[1] = evalvarn`(v) for some variable v, then the coefficients by increasing degree. An `FlxX` is a `t_POL` whose coefficients are `Flx`s.

In the following, an argument called `sv` is of the form `evalvarn`(v) for some variable number v.

7.3.16.1 Preconditioned reduction.

For faster reduction, the modulus `T` can be replaced by an extended modulus (`FlxT`) in all `Flxq`-classes functions, and in `Flx_divrem`.

`GEN Flx_get_red(GEN T, ulong p)` returns the extended modulus `eT`.

`GEN Flx_get_red_pre(GEN T, ulong p, ulong pi)` as `Flx_get_red`. We assume pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_Flx_mod(GEN eT)` returns the underlying modulus `T`.

`GEN get_Flx_var(GEN eT)` returns the variable number of the modulus.

`GEN get_Flx_degree(GEN eT)` returns the degree of the modulus.

Furthermore, `ZXT_to_FlxT` allows to convert an extended modulus for a `FpX` to an extended modulus for the corresponding `Flx`.

7.3.16.2 Basic operations.

In this section, pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`ulong Flx_lead(GEN x)` returns the leading coefficient of x as a `ulong` (return 0 for the zero polynomial).

`ulong Flx_constant(GEN x)` returns the constant coefficient of x as a `ulong` (return 0 for the zero polynomial).

`GEN Flx_red(GEN z, ulong p)` converts from `zx` with nonnegative coefficients to `Flx` (by reducing them mod `p`).

`int Flx_equal1(GEN x)` returns 1 (true) if the `Flx` x is equal to 1, 0 (false) otherwise.

`int Flx_equal(GEN x, GEN y)` returns 1 (true) if the `Flx` x and y are equal, and 0 (false) otherwise.

`GEN Flx_copy(GEN x)` returns a copy of `x`.

`GEN Flx_add(GEN x, GEN y, ulong p)`

`GEN Flx_Fl_add(GEN y, ulong x, ulong p)`

`GEN Flx_neg(GEN x, ulong p)`

`GEN Flx_neg_inplace(GEN x, ulong p)`, same as `Flx_neg`, in place (`x` is destroyed).

`GEN Flx_sub(GEN x, GEN y, ulong p)`

`GEN Flx_Fl_sub(GEN y, ulong x, ulong p)`

`GEN Flx_halve(GEN x, ulong p)` returns z such that $2z = x$ modulo p assuming such z exists.

`GEN Flx_mul(GEN x, GEN y, ulong p)`

`GEN Flx_mul_pre(GEN x, GEN y, ulong p, ulong pi)`

`GEN Flx_Fl_mul(GEN y, ulong x, ulong p)`

`GEN Flx_double(GEN y, ulong p)` returns $2y$.

`GEN Flx_triple(GEN y, ulong p)` returns $3y$.

`GEN Flx_mulu(GEN y, ulong x, ulong p)` as `Flx_Fl_mul` but do not assume that $x < p$.

`GEN Flx_Fl_mul_to_monic(GEN y, ulong x, ulong p)` returns yx assuming the result is monic of the same degree as y (in particular $x \neq 0$).

`GEN Flx_sqr(GEN x, ulong p)`

`GEN Flx_sqr_pre(GEN x, ulong p, ulong pi)`

`GEN Flx_powu(GEN x, ulong n, ulong p)` return x^n.

`GEN Flx_powu_pre(GEN x, ulong n, ulong p, ulong pi)`

`GEN Flx_divrem(GEN x, GEN y, ulong p, GEN *pr)`, here p must be prime.

`GEN Flx_divrem_pre(GEN x, GEN y, ulong p, ulong pi, GEN *pr)`

`GEN Flx_div(GEN x, GEN y, ulong p)`, here p must be prime.

`GEN Flx_div_pre(GEN x, GEN y, ulong p, ulong pi)`

`GEN Flx_rem(GEN x, GEN y, ulong p)`, here p must be prime.

`GEN Flx_rem_pre(GEN x, GEN y, ulong p, ulong pi)`

`GEN Flx_deriv(GEN z, ulong p)`

`GEN Flx_integ(GEN z, ulong p)`, here p must be prime.

`GEN Flx_translate1(GEN P, ulong p)` return $P(x+1)$, p must be prime. Asymptotically fast (quasi-linear in the degree of P).

`GEN Flx_translate1_basecase(GEN P, ulong p)` return $P(x+1)$, p need not be prime. Not asymptotically fast (quadratic in the degree of P).

`GEN zlx_translate1(GEN P, ulong p, long e)` return $P(x+1)$ modulo p^e for prime p. Asymptotically fast (quasi-linear in the degree of P).

`GEN Flx_diff1(GEN P, ulong p)` return $P(x+1) - P(x)$; p must be prime.

`GEN Flx_digits(GEN x, GEN B, ulong p)` returns a vector of `Flx` $[c_0, \ldots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^{n} c_i B^i$.

GEN FlxV_Flx_fromdigits(GEN v, GEN B, ulong p) where $v = [c_0, \ldots, c_n]$ is a vector of Flx, returns $\sum_{i=0}^{n} c_i B^i$.

GEN Flx_Frobenius(GEN T, ulong p) here p must be prime.

GEN Flx_Frobenius_pre(GEN T, ulong p, ulong pi)

GEN Flx_matFrobenius(GEN T, ulong p) here p must be prime.

GEN Flx_matFrobenius_pre(GEN T, ulong p, ulong pi)

GEN Flx_gcd(GEN a, GEN b, ulong p) returns a (not necessarily monic) greatest common divisor of x and y. Here p must be prime.

GEN Flx_gcd_pre(GEN a, GEN b, ulong p)

GEN Flx_halfgcd(GEN x, GEN y, ulong p) returns a two-by-two FlxM M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$. Assumes that p is prime.

GEN Flx_halfgcd_pre(GEN a, GEN b, ulong p)

GEN Flx_extgcd(GEN a, GEN b, ulong p, GEN *ptu, GEN *ptv), here p must be prime.

GEN Flx_extgcd_pre(GEN a, GEN b, ulong p, ulong pi, GEN *ptu, GEN *ptv)

GEN Flx_roots(GEN f, ulong p) returns the vector of roots of f (without multiplicity, as a t_VECSMALL). Assumes that p is prime.

GEN Flx_roots_pre(GEN f, ulong p, ulong pi)

ulong Flx_oneroot(GEN f, ulong p) returns one root $0 \leq r < p$ of the Flx f in $\mathbf{Z}/p\mathbf{Z}$. Return p if no root exists. Assumes that p is prime.

GEN Flx_oneroot_pre(GEN f, ulong p), as Flx_oneroot

ulong Flx_oneroot_split(GEN f, ulong p) as Flx_oneroot but assume f is totally split. Assumes that p is prime.

ulong Flx_oneroot_split_pre(GEN f, ulong p, ulong pi)

long Flx_ispower(GEN f, ulong k, ulong p, GEN *pt) return 1 if the Flx f is a k-th power, 0 otherwise. If pt is not NULL, set it to g such that $g^k = f$.

GEN Flx_factor(GEN f, ulong p) Assumes that p is prime.

GEN Flx_ddf(GEN f, ulong p) Assumes that p is prime.

GEN Flx_ddf_pre(GEN f, ulong p, ulong pi)

GEN Flx_factor_squarefree(GEN f, ulong p) returns the squarefree factorization of f modulo p. This is a vector $[u_1, \ldots, u_k]$ of pairwise coprime Flx such that $u_k \neq 1$ and $f = \prod u_i^i$. Shallow function. Assumes that p is prime.

GEN Flx_factor_squarefree_pre(GEN f, ulong p, ulong pi)

GEN Flx_mod_Xn1(GEN T, ulong n, ulong p) return T modulo $(X^n + 1, p)$. Shallow function.

GEN Flx_mod_Xnm1(GEN T, ulong n, ulong p) return T modulo $(X^n - 1, p)$. Shallow function.

GEN Flx_degfact(GEN f, ulong p) as FpX_degfact. Assumes that p is prime.

GEN Flx_factorff_irred(GEN P, GEN Q, ulong p) as FpX_factorff_irred. Assumes that p is prime.

GEN Flx_rootsff(GEN P, GEN T, ulong p) as FpX_rootsff. Assumes that p is prime.

GEN Flx_factcyclo(ulong n, ulong p, ulong m) returns the factors of the n-th cyclotomic polynomial over Fp. if $m = 1$ returns a single factor.

GEN Flx_ffisom(GEN P, GEN Q, ulong l) as FpX_ffisom. Assumes that p is prime.

7.3.16.3 Miscellaneous operations.

GEN pol0_Flx(long sv) returns a zero Flx in variable v.

GEN zero_Flx(long sv) alias for pol0_Flx

GEN pol1_Flx(long sv) returns the unit Flx in variable v.

GEN polx_Flx(long sv) returns the variable v as degree 1 Flx.

GEN polxn_Flx(long n, long sv) Returns the monomial of degree n as a Flx in variable v; assume that $n \geq 0$.

GEN monomial_Flx(ulong a, long d, long sv) returns the Flx aX^d in variable v.

GEN init_Flxq(ulong p, long n, long sv) returns an irreducible polynomial of degree $\texttt{n} > 0$ over $\mathbf{F}_p$, in variable v.

GEN Flx_normalize(GEN z, ulong p), as FpX_normalize.

GEN Flx_rescale(GEN P, ulong h, ulong p) returns $h^{\deg(P)}P(x/h)$, P is a Flx and h is a nonzero integer.

GEN random_Flx(long d, long sv, ulong p) returns a random Flx in variable v, of degree less than d.

GEN Flx_recip(GEN x), returns the reciprocal polynomial

ulong Flx_resultant(GEN a, GEN b, ulong p), returns the resultant of a and b. Assumes that p is prime.

ulong Flx_resultant_pre(GEN a, GEN b, ulong p, ulong pi)

ulong Flx_extresultant(GEN a, GEN b, ulong p, GEN *ptU, GEN *ptV) given two Flx a and b, returns their resultant and sets Bezout coefficients (if the resultant is 0, the latter are not set). Assumes that p is prime.

GEN Flx_invBarrett(GEN T, ulong p), returns the Barrett inverse M of T defined by $M(x) \times x^n T(1/x) \equiv 1 \pmod{x^{n-1}}$ where n is the degree of T. Assumes that p is prime.

GEN Flx_renormalize(GEN x, long l), as FpX_renormalize, where $\texttt{l} = \texttt{lg(x)}$, in place.

GEN Flx_shift(GEN T, long n) returns $\texttt{T} * x^n$ if $n \geq 0$, and $\texttt{T} \backslash x^{-n}$ otherwise.

long Flx_val(GEN x) returns the valuation of x, i.e. the multiplicity of the 0 root.

long Flx_valrem(GEN x, GEN *Z) as RgX_valrem, returns the valuation of x. In particular, if the valuation is 0, set *Z to x, not a copy.

GEN Flx_div_by_X_x(GEN A, ulong a, ulong p, ulong *rem), returns the Euclidean quotient of the Flx A by $X - \texttt{a}$, and sets rem to the remainder A(a).

`ulong Flx_eval(GEN x, ulong y, ulong p)`, as `FpX_eval`.

`ulong Flx_eval_pre(GEN x, ulong y, ulong p, ulong pi)`

`ulong Flx_eval_powers_pre(GEN P, GEN y, ulong p, ulong pi)`. Let y be the `t_VECSMALL` $(1, a, \ldots, a^n)$, where n is the degree of the `Flx` P, return $P(a)$.

`GEN Flx_Flv_multieval(GEN P, GEN v, ulong p)` returns the vector $[P(v[1]), \ldots, P(v[n])]$ as a `Flv`.

`ulong Flx_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of the coefficients of x and y.

`ulong Flx_dotproduct_pre(GEN x, GEN y, ulong p, ulong pi)`.

`GEN Flx_deflate(GEN P, long d)` assuming P is a polynomial of the form $Q(X^d)$, return Q.

`GEN Flx_inflate(GEN P, long d)` returns $P(X^d)$.

`GEN Flx_splitting(GEN P, long k)`, as `RgX_splitting`.

`GEN Flx_blocks(GEN P, long n, long m)`, as `RgX_blocks`.

`int Flx_is_squarefree(GEN z, ulong p)`. Assumes that p is prime.

`int Flx_is_irred(GEN f, ulong p)`, as `FpX_is_irred`. Assumes that p is prime.

`int Flx_is_totally_split(GEN f, ulong p)` returns 1 if the `Flx` `f` splits into a product of distinct linear factors, 0 otherwise. Assumes that `p` is prime.

`int Flx_is_smooth(GEN f, long r, ulong p)` return 1 if all irreducible factors of f are of degree at most r, 0 otherwise. Assumes that p is prime.

`int Flx_is_smooth_pre(GEN f, long r, ulong p, ulong pi)`

`long Flx_nbroots(GEN f, ulong p)`, as `FpX_nbroots`. Assumes that p is prime.

`long Flx_nbfact(GEN z, ulong p)`, as `FpX_nbfact`. Assumes that p is prime.

`long Flx_nbfact_pre(GEN z, ulong p, ulong pi)`

`long Flx_nbfact_Frobenius(GEN f, GEN XP, ulong p)`, as `FpX_nbfact_Frobenius`. Assumes that p is prime.

`long Flx_nbfact_Frobenius_pre(GEN f, GEN XP, ulong p, ulong pi)`

`GEN Flx_degfact(GEN f, ulong p)`, as `FpX_degfact`. Assumes that p is prime.

`GEN Flx_nbfact_by_degree(GEN z, long *nb, ulong p)` Assume that the `Flx` z is squarefree mod the prime p. Returns a `t_VECSMALL` D with $\deg z$ entries, such that $D[i]$ is the number of irreducible factors of degree i. Set `nb` to the total number of irreducible factors (the sum of the $D[i]$). Assumes that p is prime.

`void Flx_ffintersect(GEN P, GEN Q, long n, ulong p, GEN*SP, GEN*SQ, GEN MA, GEN MB)`
,
as `FpX_ffintersect`. Assumes that p is prime.

`GEN Flx_Laplace(GEN x, ulong p)`

`GEN Flx_invLaplace(GEN x, ulong p)`

`GEN Flx_Newton(GEN x, long n, ulong p)`

`GEN Flx_fromNewton(GEN x, ulong p)`

`GEN Flx_Teichmuller(GEN P, ulong p, long n)` Return a `ZX` Q such that $P \equiv Q \pmod p$ and $Q(X^p) = 0 \pmod{Q, p^n}$. Assumes that p is prime.

`GEN Flv_polint(GEN x, GEN y, ulong p, long sv)` as `FpV_polint`, returning an `Flx` in variable v. Assumes that p is prime.

`GEN Flv_Flm_polint(GEN x, GEN V, ulong p, long sv)` equivalent (but faster) to applying `Flv_polint(x,...)` to all the elements of the vector V (thus, returns a `FlxV`). Assumes that p is prime.

`GEN Flv_invVandermonde(GEN L, ulong d, ulong p)` L being a `Flv` of length n, return the inverse M of the Vandermonde matrix attached to the elements of L, multiplied by `d`. If A is a `Flv` and $B = MA$, then the polynomial $P = \sum_{i=1}^{n} B[i]X^{i-1}$ verifies $P(L[i]) = dA[i]$ for $1 \leq i \leq n$. Assumes that p is prime.

`GEN Flv_roots_to_pol(GEN a, ulong p, long sv)` as `FpV_roots_to_pol` returning an `Flx` in variable v.

7.3.17 `FlxV`. See `FpXV` operations.

`GEN FlxV_Flc_mul(GEN V, GEN W, ulong p)`, as `FpXV_FpC_mul`.

`GEN FlxV_red(GEN V, ulong p)` reduces each components with `Flx_red`.

`GEN FlxV_prod(GEN V, ulong p)`, `V` being a vector of `Flx`, returns their product.

`ulong FlxC_eval_powers_pre(GEN x, GEN y, ulong p, ulong pi)` apply `Flx_eval_powers_pre` to all elements of `x`.

`GEN FlxV_Flv_multieval(GEN F, GEN v, ulong p)` assuming F is a vector of `Flx` with m entries and v is a `Flv` with m entries, returns the n-components vector (`FlvV`) whose j-th entry is $[F_j(v[1]), \ldots, F_j(v[n])]$, with $F_j = F[j]$.

`GEN FlxC_neg(GEN x, ulong p)`

`GEN FlxC_sub(GEN x, GEN y, ulong p)`

`GEN zero_FlxC(long n, long sv)`

7.3.18 `FlxM`. See `FpXM` operations.

`ulong FlxM_eval_powers_pre(GEN M, GEN y, ulong p, ulong pi)` this function applies `FlxC_eval_powers_pre` to all entries of `M`.

`GEN FlxM_neg(GEN x, ulong p)`

`GEN FlxM_sub(GEN x, GEN y, ulong p)`

`GEN zero_FlxM(long r, long c, long sv)`

7.3.19 `FlxT`. See `FpXT` operations.

`GEN FlxT_red(GEN V, ulong p)` reduces each leaf with `Flx_red`.

7.3.20 `Flxn`. See `FpXn` operations. In this section, pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flxn_mul(GEN a, GEN b, long n, ulong p)` returns ab modulo X^n.

`GEN Flxn_mul_pre(GEN a, GEN b, long n, ulong p, ulong pi)`

`GEN Flxn_sqr(GEN a, long n, ulong p)` returns a^2 modulo X^n.

`GEN Flxn_sqr_pre(GEN a, long n, ulong p, ulong pi)`

`GEN Flxn_inv(GEN a, long n, ulong p)` returns $1/a$ modulo X^n.

`GEN Flxn_div(GEN a, GEN b, long n, ulong p)` returns a/b modulo X^n.

`GEN Flxn_div_pre(GEN a, GEN b, long n, ulong p, ulong pi)`

`GEN Flxn_red(GEN a, long n)` returns a modulo X^n.

`GEN Flxn_exp(GEN x, long n, ulong p)` return $\exp(x)$ as a composition of formal power series. It is required that the valuation of x is positive and that $p > n$.

`GEN Flxn_expint(GEN f, long n, ulong p)` return $\exp(F)$ where F is the primitive of f that vanishes at 0. It is required that $p > n$.

7.3.21 `Flxq`. See `FpXQ` operations. In this section, pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN Flxq_add(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_sub(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_mul(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_mul_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN Flxq_sqr(GEN y, GEN T, ulong p)`

`GEN Flxq_sqr_pre(GEN y, GEN T, ulong p)`

`GEN Flxq_inv(GEN x, GEN T, ulong p)`

`GEN Flxq_inv_pre(GEN x, GEN T, ulong p, ulong pi)`

`GEN Flxq_invsafe(GEN x, GEN T, ulong p)`

`GEN Flxq_invsafe_pre(GEN x, GEN T, ulong p, ulong pi)`

`GEN Flxq_div(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_div_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN Flxq_pow(GEN x, GEN n, GEN T, ulong p)`

`GEN Flxq_pow_pre(GEN x, GEN n, GEN T, ulong p, ulong pi)`

`GEN Flxq_powu(GEN x, ulong n, GEN T, ulong p)`

`GEN Flxq_powu_pre(GEN x, ulong n, GEN T, ulong p)`

`GEN FlxqV_factorback(GEN L, GEN e, GEN Tp, ulong p)`

`GEN Flxq_pow_init(GEN x, GEN n, long k, GEN T, ulong p)`

GEN Flxq_pow_init_pre(GEN x, GEN n, long k, GEN T, ulong p, ulong pi)

GEN Flxq_pow_table(GEN R, GEN n, GEN T, ulong p)

GEN Flxq_pow_table_pre(GEN R, GEN n, GEN T, ulong p, ulong pi)

GEN Flxq_powers(GEN x, long n, GEN T, ulong p)

GEN Flxq_powers_pre(GEN x, long n, GEN T, ulong p, ulong pi)

GEN Flxq_matrix_pow(GEN x, long m, long n, GEN T, ulong p), see FpXQ_matrix_pow.

GEN Flxq_matrix_pow_pre(GEN x, long m, long n, GEN T, ulong p, ulong pi)

GEN Flxq_autpow(GEN a, long n, GEN T, ulong p) see FpXQ_autpow.

GEN Flxq_autpow_pre(GEN a, long n, GEN T, ulong p, ulong pi)

GEN Flxq_autpowers(GEN a, long n, GEN T, ulong p) return $[X, \sigma(X), \ldots, \sigma^n(X)]$, assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_p[X]/T(X)$.

GEN Flxq_autsum(GEN a, long n, GEN T, ulong p) see FpXQ_autsum.

GEN Flxq_auttrace(GEN a, ulong n, GEN T, ulong p) see FpXQ_auttrace.

GEN Flxq_auttrace_pre(GEN a, ulong n, GEN T, ulong p, ulong pi)

GEN Flxq_ffisom_inv(GEN S, GEN T, ulong p), as FpXQ_ffisom_inv.

GEN Flx_Flxq_eval(GEN f, GEN x, GEN T, ulong p) returns f(x).

GEN Flx_Flxq_eval_pre(GEN f, GEN x, GEN T, ulong p, ulong pi)

GEN Flx_FlxqV_eval(GEN f, GEN x, GEN T, ulong p), see FpX_FpXQV_eval.

GEN Flx_FlxqV_eval_pre(GEN f, GEN x, GEN T, ulong p, ulong pi)

GEN FlxC_Flxq_eval(GEN C, GEN x, GEN T, ulong p), see FpXC_FpXQ_eval.

GEN FlxC_Flxq_eval_pre(GEN C, GEN x, GEN T, ulong p, ulong pi)

GEN FlxC_FlxqV_eval(GEN C, GEN V, GEN T, ulong p) see FpXC_FpXQV_eval.

GEN FlxC_FlxqV_eval_pre(GEN C, GEN V, GEN T, ulong p, ulong pi)

GEN FlxqV_roots_to_pol(GEN V, GEN T, ulong p, long v) as FqV_roots_to_pol returning an FlxqX in variable v.

int Flxq_issquare(GEN x, GEN T, ulong p) returns 1 if x is a square and 0 otherwise. Assume that T is irreducible mod p.

int Flxq_is2npower(GEN x, long n, GEN T, ulong p) returns 1 if x is a 2^n-th power and 0 otherwise. Assume that T is irreducible mod p.

GEN Flxq_order(GEN a, GEN ord, GEN T, ulong p) as FpXQ_order.

GEN Flxq_log(GEN a, GEN g, GEN ord, GEN T, ulong p) as FpXQ_log

GEN Flxq_sqrtn(GEN x, GEN n, GEN T, ulong p, GEN *zn) as FpXQ_sqrtn.

GEN Flxq_sqrt(GEN x, GEN T, ulong p) returns a square root of x. Return NULL if x is not a square.

GEN Flxq_lroot(GEN a, GEN T, ulong p) returns x such that $x^p = a$.

GEN Flxq_lroot_pre(GEN a, GEN T, ulong p, ulong pi)

GEN Flxq_lroot_fast(GEN a, GEN V, GEN T, ulong p) assuming that V=Flxq_powers(s,p-1,T,p) where $s(x)^p \equiv x \pmod{T(x), p}$, returns b such that $b^p = a$. Only useful if p is less than the degree of T.

GEN Flxq_lroot_fast_pre(GEN a, GEN V, GEN T, ulong p, ulong pi)

GEN Flxq_charpoly(GEN x, GEN T, ulong p) returns the characteristic polynomial of x

GEN Flxq_minpoly(GEN x, GEN T, ulong p) returns the minimal polynomial of x

GEN Flxq_minpoly_pre(GEN x, GEN T, ulong p, ulong pi)

ulong Flxq_norm(GEN x, GEN T, ulong p) returns the norm of x

ulong Flxq_trace(GEN x, GEN T, ulong p) returns the trace of x

GEN Flxq_conjvec(GEN x, GEN T, ulong p) returns the conjugates $[x, x^p, x^{p^2}, \ldots, x^{p^{n-1}}]$ where n is the degree of T.

GEN gener_Flxq(GEN T, ulong p, GEN *po) returns a primitive root modulo (T, p). T is an Flx assumed to be irreducible modulo the prime p. If po is not NULL it is set to $[o, fa]$, where o is the order of the multiplicative group of the finite field, and fa is its factorization.

7.3.22 FlxX. See FpXX operations. In this section, we assume pi is the pseudoinverse of p, or 0 in which case we assume SMALL_ULONG(p).

GEN pol1_FlxX(long vX, long sx) returns the unit FlxX as a t_POL in variable vX which only coefficient is pol1_Flx(sx).

GEN polx_FlxX(long vX, long sx) returns the variable X as a degree 1 t_POL with Flx coefficients in the variable x.

long FlxY_degreex(GEN P) return the degree of P with respect to the secondary variable.

GEN FlxX_add(GEN P, GEN Q, ulong p)

GEN FlxX_sub(GEN P, GEN Q, ulong p)

GEN FlxX_Fl_mul(GEN x, ulong y, ulong p)

GEN FlxX_double(GEN x, ulong p)

GEN FlxX_triple(GEN x, ulong p)

GEN FlxX_neg(GEN x, ulong p)

GEN FlxX_Flx_add(GEN x, GEN y, ulong p)

GEN FlxX_Flx_sub(GEN x, GEN y, ulong p)

GEN FlxX_Flx_mul(GEN x, GEN y, ulong p)

GEN FlxY_Flx_div(GEN x, GEN y, ulong p) divides the coefficients of x by y using Flx_div.

GEN FlxX_deriv(GEN P, ulong p) returns the derivative of P with respect to the main variable.

GEN FlxX_Laplace(GEN x, ulong p)

GEN FlxX_invLaplace(GEN x, ulong p)

`GEN FlxY_evalx(GEN P, ulong z, ulong p)` P being an `FlxY`, returns the `Flx` $P(z,Y)$, where Y is the main variable of P.

`GEN FlxY_evalx_pre(GEN P, ulong z, ulong p, ulong pi)`

`GEN FlxX_translate1(GEN P, ulong p, long n)` P being an `FlxX` with all coefficients of degree at most n, return $(P(x,Y+1)$, where Y is the main variable of P.

`GEN zlxX_translate1(GEN P, ulong p, long e, long n)` P being an `zlxX` with all coefficients of degree at most n, return $(P(x,Y+1)$ modulo p^e for prime p , where Y is the main variable of P.

`GEN FlxY_Flx_translate(GEN P, GEN f, ulong p)` P being an `FlxY` and f being an `Flx`, return $(P(x,Y+f(x))$, where Y is the main variable of P.

`ulong FlxY_evalx_powers_pre(GEN P, GEN xp, ulong p, ulong pi)`, `xp` being the vector $[1,x,\ldots,x^n]$, where n is larger or equal to the degree of P in X, return $P(x,Y)$, where Y is the main variable of Q.

`ulong FlxY_eval_powers_pre(GEN P, GEN xp, GEN yp, ulong p, ulong pi)`, `xp` being the vector $[1,x,\ldots,x^n]$, where n is larger or equal to the degree of P in X and `yp` being the vector $[1,y,\ldots,y^m]$, where m is larger or equal to the degree of P in Y return $P(x,y)$.

`GEN FlxY_Flxq_evalx(GEN x, GEN y, GEN T, ulong p)` as `FpXY_FpXQ_evalx`.

`GEN FlxY_Flxq_evalx_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN FlxY_FlxqV_evalx(GEN x, GEN V, GEN T, ulong p)` as `FpXY_FpXQV_evalx`.

`GEN FlxY_FlxqV_evalx_pre(GEN x, GEN V, GEN T, ulong p, ulong pi)`

`GEN FlxX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg(x)`, in place.

`GEN FlxX_resultant(GEN u, GEN v, ulong p, long sv)` Returns $\mathrm{Res}_X(u,v)$, which is an `Flx`. The coefficients of `u` and `v` are assumed to be in the variable v.

`GEN Flx_FlxY_resultant(GEN a, GEN b, ulong p)` Returns $\mathrm{Res}_x(a,b)$, which is an `Flx` in the main variable of `b`.

`GEN FlxX_blocks(GEN P, long n, long m, long sv)`, as `RgX_blocks`, where v is the secondary variable.

`GEN FlxX_shift(GEN a, long n, long sv)`, as `RgX_shift_shallow`, where v is the secondary variable.

`GEN FlxX_swap(GEN x, long n, long ws)`, as `RgXY_swap`.

`GEN FlxYqq_pow(GEN x, GEN n, GEN S, GEN T, ulong p)`, as `FpXYQQ_pow`.

7.3.23 `FlxXV`, `FlxXC`, `FlxXM`. See `FpXX` operations.

`GEN FlxXC_sub(GEN x, GEN y, ulong p)`

7.3.24 `FlxqX`. See `FpXQX` operations.

7.3.24.1 Preconditioned reduction.

For faster reduction, the modulus `S` can be replaced by an extended modulus, which is an `FlxqXT`, in all `FlxqXQ`-classes functions, and in `FlxqX_rem` and `FlxqX_divrem`.

`GEN FlxqX_get_red(GEN S, GEN T, ulong p)` returns the extended modulus `eS`.

`GEN FlxqX_get_red_pre(GEN S, GEN T, ulong p, ulong pi)`, where pi is a pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_FlxqX_mod(GEN eS)` returns the underlying modulus `S`.

`GEN get_FlxqX_var(GEN eS)` returns the variable number of the modulus.

`GEN get_FlxqX_degree(GEN eS)` returns the degree of the modulus.

7.3.24.2 basic functions.

In this section, pi is a pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN random_FlxqX(long d, long v, GEN T, ulong p)` returns a random `FlxqX` in variable `v`, of degree less than `d`.

`GEN zxX_to_Kronecker(GEN P, GEN Q)` assuming $P(X,Y)$ is a polynomial of degree in X strictly less than n, returns $P(X, X^{2*n-1})$, the Kronecker form of P.

`GEN Kronecker_to_FlxqX(GEN z, GEN T, ulong p)`. Let $n = \deg T$ and let $P(X,Y) \in \mathbf{Z}[X,Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{F}_p[X]/(T)$ and $\deg_X P < 2n-1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P, this function returns $Q \in \mathbf{Z}[X,t]$ such that Q is congruent to $P(X,t)$ mod $(p, T(X))$, $\deg_X Q < n$, and all coefficients are in $[0,p[$. Not stack-clean. Note that t need not be the same variable as Y!

`GEN Kronecker_to_FlxqX_pre(GEN z, GEN T, ulong p, ulong pi)`

`GEN FlxqX_red(GEN z, GEN T, ulong p)`

`GEN FlxqX_red_pre(GEN z, GEN T, ulong p, ulong pi)`

`GEN FlxqX_normalize(GEN z, GEN T, ulong p)`

`GEN FlxqX_normalize_pre(GEN z, GEN T, ulong p, ulong pi)`

`GEN FlxqX_mul(GEN x, GEN y, GEN T, ulong p)`

`GEN FlxqX_mul_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN FlxqX_Flxq_mul(GEN P, GEN U, GEN T, ulong p)`

`GEN FlxqX_Flxq_mul_pre(GEN P, GEN U, GEN T, ulong p, ulong pi)`

`GEN FlxqX_Flxq_mul_to_monic(GEN P, GEN U, GEN T, ulong p)` returns $P*U$ assuming the result is monic of the same degree as P (in particular $U \neq 0$).

`GEN FlxqX_Flxq_mul_to_monic_pre(GEN P, GEN U, GEN T, ulong p, ulong pi)`

`GEN FlxqX_sqr(GEN x, GEN T, ulong p)`

GEN FlxqX_sqr_pre(GEN x, GEN T, ulong p, ulong pi)

GEN FlxqX_powu(GEN x, ulong n, GEN T, ulong p)

GEN FlxqX_powu_pre(GEN x, ulong n, GEN T, ulong p, ulong pi)

GEN FlxqX_divrem(GEN x, GEN y, GEN T, ulong p, GEN *pr)

GEN FlxqX_divrem_pre(GEN x, GEN y, GEN T, ulong p, ulong pi, GEN *pr)

GEN FlxqX_div(GEN x, GEN y, GEN T, ulong p)

GEN FlxqX_div_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxqX_rem(GEN x, GEN y, GEN T, ulong p)

GEN FlxqX_rem_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxqX_invBarrett(GEN T, GEN Q, ulong p)

GEN FlxqX_invBarrett_pre(GEN T, GEN Q, ulong p, ulong pi)

GEN FlxqX_gcd(GEN x, GEN y, ulong p) returns a (not necessarily monic) greatest common divisor of x and y.

GEN FlxqX_gcd_pre(GEN x, GEN y, ulong p, ulong pi)

GEN FlxqX_extgcd(GEN x, GEN y, GEN T, ulong p, GEN *ptu, GEN *ptv)

GEN FlxqX_extgcd_pre(GEN x, GEN y, GEN T, ulong p, ulong pi, GEN *ptu, GEN *ptv)

GEN FlxqX_halfgcd(GEN x, GEN y, GEN T, ulong p), see FpX_halfgcd.

GEN FlxqX_halfgcd_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxqX_resultant(GEN x, GEN y, GEN T, ulong p)

GEN FlxqX_saferesultant(GEN P, GEN Q, GEN T, ulong p) Returns the resultant of P and Q if Euclid's algorithm succeeds and NULL otherwise. In particular, if p is not prime or T is not irreducible over $\mathbf{F}_p[X]$, the routine may still be used (but will fail if noninvertible leading terms occur).

GEN FlxqX_disc(GEN x, GEN T, ulong p)

GEN FlxqXV_prod(GEN V, GEN T, ulong p)

GEN FlxqX_safegcd(GEN P, GEN Q, GEN T, ulong p) Returns the *monic* GCD of P and Q if Euclid's algorithm succeeds and NULL otherwise. In particular, if p is not prime or T is not irreducible over $\mathbf{F}_p[X]$, the routine may still be used (but will fail if noninvertible leading terms occur).

GEN FlxqX_dotproduct(GEN x, GEN y, GEN T, ulong p) returns the scalar product of the coefficients of x and y.

GEN FlxqX_Newton(GEN x, long n, GEN T, ulong p)

GEN FlxqX_Newton_pre(GEN x, long n, GEN T, ulong p, ulong pi)

GEN FlxqX_fromNewton(GEN x, GEN T, ulong p)

GEN FlxqX_fromNewton_pre(GEN x, GEN T, ulong p, ulong pi) We assume *pi* is a pseudoinverse of p, or 0 in which case we assume SMALL_ULONG(p).

long FlxqX_is_squarefree(GEN S, GEN T, ulong p), as FpX_is_squarefree.

long FlxqX_ispower(GEN f, ulong k, GEN T, ulong p, GEN *pt) return 1 if the FlxqX f is a k-th power, 0 otherwise. If pt is not NULL, set it to g such that $g^k = f$.

GEN FlxqX_Frobenius(GEN S, GEN T, ulong p), as FpXQX_Frobenius

GEN FlxqX_Frobenius_pre(GEN S, GEN T, ulong p, ulong pi)

GEN FlxqX_roots(GEN f, GEN T, ulong p) return the roots of f in $\mathbf{F}_p[X]/(T)$. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FlxqX_factor(GEN f, GEN T, ulong p) return the factorization of f over $\mathbf{F}_p[X]/(T)$. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FlxqX_factor_squarefree(GEN f, GEN T, ulong p) returns the squarefree factorization of f, see FpX_factor_squarefree.

GEN FlxqX_factor_squarefree_pre(GEN f, GEN T, ulong p, ulong pi)

GEN FlxqX_ddf(GEN f, GEN T, ulong p) as FpX_ddf.

long FlxqX_ddf_degree(GEN f, GEN XP, GEN T, GEN p), as FpX_ddf_degree.

GEN FlxqX_degfact(GEN f, GEN T, ulong p), as FpX_degfact.

long FlxqX_nbroots(GEN S, GEN T, ulong p), as FpX_nbroots.

long FlxqX_nbfact(GEN S, GEN T, ulong p), as FpX_nbfact.

long FlxqX_nbfact_Frobenius(GEN S, GEN Xq, GEN T, ulong p), as FpX_nbfact_Frobenius.

GEN FlxqX_nbfact_by_degree(GEN z, long *nb, GEN T, ulong p) Assume that the FlxqX z is squarefree mod the prime p. Returns a t_VECSMALL D with $\deg z$ entries, such that $D[i]$ is the number of irreducible factors of degree i. Set nb to the total number of irreducible factors (the sum of the $D[i]$).

GEN FlxqX_FlxqXQ_eval(GEN Q, GEN x, GEN S, GEN T, ulong p) as FpX_FpXQ_eval.

GEN FlxqX_FlxqXQ_eval_pre(GEN Q, GEN x, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqX_FlxqXQV_eval(GEN P, GEN V, GEN S, GEN T, ulong p) as FpX_FpXQV_eval.

GEN FlxqX_FlxqXQV_eval_pre(GEN P, GEN V, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXC_FlxqXQ_eval(GEN Q, GEN x, GEN S, GEN T, ulong p) as FpXC_FpXQ_eval.

GEN FlxqXC_FlxqXQ_eval_pre(GEN Q, GEN x, GEN S, GEN T, ulong p, ulong pi)

GEN FlxqXC_FlxqXQV_eval(GEN P, GEN V, GEN S, GEN T, ulong p) as FpXC_FpXQV_eval.

GEN FlxqXC_FlxqXQV_eval_pre(GEN P, GEN V, GEN S, GEN T, ulong p, ulong pi)

7.3.25 `FlxqXQ`. See `FpXQXQ` operations. In this section, pi is a pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN FlxqXQ_mul(GEN x, GEN y, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_mul_pre(GEN x, GEN y, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_sqr(GEN x, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_sqr_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_inv(GEN x, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_inv_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_invsafe(GEN x, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_invsafe_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_div(GEN x, GEN y, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_div_pre(GEN x, GEN y, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_pow(GEN x, GEN n, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_pow_pre(GEN x, GEN n, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_powu(GEN x, ulong n, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_powu_pre(GEN x, ulong n, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_powers(GEN x, long n, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_powers_pre(GEN x, long n, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_matrix_pow(GEN x, long n, long m, GEN S, GEN T, ulong p)`

`GEN FlxqXQ_autpow(GEN a, long n, GEN S, GEN T, ulong p)` as `FpXQXQ_autpow`

`GEN FlxqXQ_autpow_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_autsum(GEN a, long n, GEN S, GEN T, ulong p)` as `FpXQXQ_autsum`

`GEN FlxqXQ_autsum_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_auttrace(GEN a, long n, GEN S, GEN T, ulong p)` as `FpXQXQ_auttrace`

`GEN FlxqXQ_auttrace_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)`

`GEN FlxqXQ_halfFrobenius(GEN A, GEN S, GEN T, ulong p)`, as `FpXQXQ_halfFrobenius`

`GEN FlxqXQ_minpoly(GEN x, GEN S, GEN T, ulong p)`, as `FpXQ_minpoly`

`GEN FlxqXQ_minpoly_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)`

7.3.26 `FlxqXn`. See `FpXn` operations. In this section, we assume pi is the pseudoinverse of p, or 0 in which case we assume `SMALL_ULONG`(p).

`GEN FlxXn_red(GEN a, long n)` returns a modulo X^n.

`GEN FlxqXn_mul(GEN a, GEN b, long n, GEN T, ulong p)`

`GEN FlxqXn_mul_pre(GEN a, GEN b, long n, GEN T, ulong p, ulong pi)`

`GEN FlxqXn_sqr(GEN a, long n, GEN T, ulong p)`

`GEN FlxqXn_sqr_pre(GEN a, long n, GEN T, ulong p, ulong pi)`

`GEN FlxqXn_inv(GEN a, long n, GEN T, ulong p)`

`GEN FlxqXn_inv_pre(GEN a, long n, GEN T, ulong p, ulong pi)`

`GEN FlxqXn_expint(GEN a, long n, GEN T, ulong p)`

`GEN FlxqXn_expint_pre(GEN a, long n, GEN T, ulong p, ulong pi)`

7.3.27 `F2x`. An `F2x` `z` is a `t_VECSMALL` representing a polynomial over $\mathbf{F}_2[X]$. Specifically `z[0]` is the usual codeword, `z[1] = evalvarn`(v) for some variable v and the coefficients are given by the bits of remaining words by increasing degree.

7.3.27.1 Preconditioned reduction.

For faster reduction, the modulus `T` can be replaced by an extended modulus (`FlxT`) in all `Flxq`-classes functions, and in `Flx_divrem`.

`GEN F2x_get_red(GEN T)` returns the extended modulus `eT`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_F2x_mod(GEN eT)` returns the underlying modulus `T`.

`GEN get_F2x_var(GEN eT)` returns the variable number of the modulus.

`GEN get_F2x_degree(GEN eT)` returns the degree of the modulus.

7.3.27.2 Basic operations.

`ulong F2x_coeff(GEN x, long i)` returns the coefficient $i \geq 0$ of x.

`void F2x_clear(GEN x, long i)` sets the coefficient $i \geq 0$ of x to 0.

`void F2x_flip(GEN x, long i)` adds 1 to the coefficient $i \geq 0$ of x.

`void F2x_set(GEN x, long i)` sets the coefficient $i \geq 0$ of x to 1.

`GEN F2x_copy(GEN x)`

`GEN Flx_to_F2x(GEN x)`

`GEN Z_to_F2x(GEN x, long sv)`

`GEN ZX_to_F2x(GEN x)`

`GEN F2v_to_F2x(GEN x, long sv)`

`GEN F2x_to_Flx(GEN x)`

GEN F2x_to_F2xX(GEN x, long sv)

GEN F2x_to_ZX(GEN x)

GEN pol0_F2x(long sv) returns a zero F2x in variable v.

GEN zero_F2x(long sv) alias for pol0_F2x.

GEN pol1_F2x(long sv) returns the F2x in variable v constant to 1.

GEN polx_F2x(long sv) returns the variable v as degree 1 F2x.

GEN monomial_F2x(long d, long sv) returns the F2x X^d in variable v.

GEN random_F2x(long d, long sv) returns a random F2x in variable v, of degree less than d.

long F2x_degree(GEN x) returns the degree of the F2x x. The degree of 0 is defined as -1.

GEN F2x_recip(GEN x)

int F2x_equal1(GEN x)

int F2x_equal(GEN x, GEN y)

GEN F2x_1_add(GEN y) returns y+1 where y is a Flx.

GEN F2x_add(GEN x, GEN y)

GEN F2x_mul(GEN x, GEN y)

GEN F2x_sqr(GEN x)

GEN F2x_divrem(GEN x, GEN y, GEN *pr)

GEN F2x_rem(GEN x, GEN y)

GEN F2x_div(GEN x, GEN y)

GEN F2x_renormalize(GEN x, long lx)

GEN F2x_deriv(GEN x)

GEN F2x_deflate(GEN x, long d)

ulong F2x_eval(GEN P, ulong u) returns $P(u)$.

void F2x_shift(GEN x, long d) as RgX_shift

void F2x_even_odd(GEN P, GEN *pe, GEN *po) as RgX_even_odd

long F2x_valrem(GEN x, GEN *Z)

GEN F2x_extgcd(GEN a, GEN b, GEN *ptu, GEN *ptv)

GEN F2x_gcd(GEN a, GEN b)

GEN F2x_halfgcd(GEN a, GEN b)

int F2x_issquare(GEN x) returns 1 if x is a square of a F2x and 0 otherwise.

int F2x_is_irred(GEN f), as FpX_is_irred.

GEN F2x_degfact(GEN f) as FpX_degfact.

GEN F2x_sqrt(GEN x) returns the squareroot of x, assuming x is a square of a F2x.

GEN F2x_Frobenius(GEN T)

GEN F2x_matFrobenius(GEN T)

GEN F2x_factor(GEN f)

GEN F2x_factor_squarefree(GEN f)

GEN F2x_ddf(GEN f)

GEN F2x_Teichmuller(GEN P, long n) Return a ZX Q such that $P \equiv Q \pmod 2$ and $Q(X^p) = 0 \pmod{Q, 2^n}$.

7.3.28 F2xq. See FpXQ operations.

GEN F2xq_mul(GEN x, GEN y, GEN T)

GEN F2xq_sqr(GEN x, GEN T)

GEN F2xq_div(GEN x, GEN y, GEN T)

GEN F2xq_inv(GEN x, GEN T)

GEN F2xq_invsafe(GEN x, GEN T)

GEN F2xq_pow(GEN x, GEN n, GEN T)

GEN F2xq_powu(GEN x, ulong n, GEN T)

GEN F2xq_pow_init(GEN x, GEN n, long k, GEN T)

GEN F2xq_pow_table(GEN R, GEN n, GEN T)

ulong F2xq_trace(GEN x, GEN T)

GEN F2xq_conjvec(GEN x, GEN T) returns the vector of conjugates $[x, x^2, x^{2^2}, \ldots, x^{2^{n-1}}]$ where n is the degree of T.

GEN F2xq_log(GEN a, GEN g, GEN ord, GEN T)

GEN F2xq_order(GEN a, GEN ord, GEN T)

GEN F2xq_Artin_Schreier(GEN a, GEN T) returns a solution of $x^2 + x = a$, assuming it exists.

GEN F2xq_sqrt(GEN a, GEN T)

GEN F2xq_sqrt_fast(GEN a, GEN s, GEN T) assuming that $s^2 \equiv x \pmod{T(x)}$, computes $b \equiv a(s) \pmod T$ so that $b^2 = a$.

GEN F2xq_sqrtn(GEN a, GEN n, GEN T, GEN *zeta)

GEN gener_F2xq(GEN T, GEN *po)

GEN F2xq_powers(GEN x, long n, GEN T)

GEN F2xq_matrix_pow(GEN x, long m, long n, GEN T)

GEN F2x_F2xq_eval(GEN f, GEN x, GEN T)

GEN F2x_F2xqV_eval(GEN f, GEN x, GEN T), see FpX_FpXQV_eval.

GEN F2xq_autpow(GEN a, long n, GEN T) computes $\sigma^n(X)$ assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_2[X]/T(X)$.

7.3.29 F2xn. See FpXn operations.

GEN F2xn_red(GEN a, long n)

GEN F2xn_div(GEN x, GEN y, long e)

GEN F2xn_inv(GEN x, long e)

7.3.30 F2xqV, F2xqM.. See FqV, FqM operations.

GEN F2xqM_F2xqC_gauss(GEN a, GEN b, GEN T)

GEN F2xqM_F2xqC_invimage(GEN a, GEN b, GEN T)

GEN F2xqM_F2xqC_mul(GEN a, GEN b, GEN T)

GEN F2xqM_deplin(GEN x, GEN T)

GEN F2xqM_det(GEN a, GEN T)

GEN F2xqM_gauss(GEN a, GEN b, GEN T)

GEN F2xqM_image(GEN x, GEN T)

GEN F2xqM_indexrank(GEN x, GEN T)

GEN F2xqM_inv(GEN a, GEN T)

GEN F2xqM_invimage(GEN a, GEN b, GEN T)

GEN F2xqM_ker(GEN x, GEN T)

GEN F2xqM_mul(GEN a, GEN b, GEN T)

long F2xqM_rank(GEN x, GEN T)

GEN F2xqM_suppl(GEN x, GEN T)

GEN matid_F2xqM(long n, GEN T)

7.3.31 F2xX.. See FpXX operations.

GEN ZXX_to_F2xX(GEN x, long v)

GEN FlxX_to_F2xX(GEN x)

GEN F2xX_to_FlxX(GEN B)

GEN F2xX_to_F2xC(GEN B, long N, long sv)

GEN F2xXV_to_F2xM(GEN B, long N, long sv)

GEN F2xX_to_ZXX(GEN B)

GEN F2xX_renormalize(GEN x, long lx)

long F2xY_degreex(GEN P) return the degree of P with respect to the secondary variable.

GEN pol1_F2xX(long v, long sv)

GEN polx_F2xX(long v, long sv)

GEN F2xX_add(GEN x, GEN y)

GEN F2xX_F2x_add(GEN x, GEN y)

GEN F2xX_F2x_mul(GEN x, GEN y)

GEN F2xX_deriv(GEN P) returns the derivative of P with respect to the main variable.

GEN Kronecker_to_F2xqX(GEN z, GEN T)

GEN F2xX_to_Kronecker(GEN z, GEN T)

GEN F2xY_F2xq_evalx(GEN x, GEN y, GEN T) as FpXY_FpXQ_evalx.

GEN F2xY_F2xqV_evalx(GEN x, GEN V, GEN T) as FpXY_FpXQV_evalx.

7.3.32 F2xXV/F2xXC.. See FpXXV operations.

GEN FlxXC_to_F2xXC(GEN B)

GEN F2xXC_to_ZXXC(GEN B)

7.3.33 F2xqX.. See FlxqX operations.

7.3.33.1 Preconditioned reduction.

For faster reduction, the modulus S can be replaced by an extended modulus, which is an F2xqXT, in all F2xqXQ-classes functions, and in F2xqX_rem and F2xqX_divrem.

GEN F2xqX_get_red(GEN S, GEN T) returns the extended modulus eS.

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get_F2xqX_mod(GEN eS) returns the underlying modulus S.

GEN get_F2xqX_var(GEN eS) returns the variable number of the modulus.

GEN get_F2xqX_degree(GEN eS) returns the degree of the modulus.

7.3.33.2 basic functions.

GEN random_F2xqX(long d, long v, GEN T, ulong p) returns a random F2xqX in variable v, of degree less than d.

GEN F2xqX_red(GEN z, GEN T)

GEN F2xqX_normalize(GEN z, GEN T)

GEN F2xqX_F2xq_mul(GEN P, GEN U, GEN T)

GEN F2xqX_F2xq_mul_to_monic(GEN P, GEN U, GEN T)

GEN F2xqX_mul(GEN x, GEN y, GEN T)

GEN F2xqX_sqr(GEN x, GEN T)

GEN F2xqX_powu(GEN x, ulong n, GEN T)

GEN F2xqX_rem(GEN x, GEN y, GEN T)

GEN F2xqX_div(GEN x, GEN y, GEN T)

GEN F2xqX_divrem(GEN x, GEN y, GEN T, GEN *pr)

GEN F2xqXQ_inv(GEN x, GEN S, GEN T)

GEN F2xqXQ_invsafe(GEN x, GEN S, GEN T)

GEN F2xqX_invBarrett(GEN T, GEN Q)

GEN F2xqX_extgcd(GEN x, GEN y, GEN T, GEN *ptu, GEN *ptv)

GEN F2xqX_gcd(GEN x, GEN y, GEN T)

GEN F2xqX_halfgcd(GEN x, GEN y, GEN T)

GEN F2xqX_resultant(GEN x, GEN y, GEN T)

GEN F2xqX_disc(GEN x, GEN T)

long F2xqX_ispower(GEN f, ulong k, GEN T, GEN *pt)

GEN F2xqX_F2xqXQ_eval(GEN Q, GEN x, GEN S, GEN T) as FpX_FpXQ_eval.

GEN F2xqX_F2xqXQV_eval(GEN P, GEN V, GEN S, GEN T) as FpX_FpXQV_eval.

GEN F2xqX_roots(GEN f, GEN T) return the roots of f in $\mathbf{F}_2[X]/(T)$. Assumes T irreducible in $\mathbf{F}_2[X]$.

GEN F2xqX_factor(GEN f, GEN T) return the factorization of f over $\mathbf{F}_2[X]/(T)$. Assumes T irreducible in $\mathbf{F}_2[X]$.

GEN F2xqX_factor_squarefree(GEN f, GEN T) as FlxqX_factor_squarefree.

GEN F2xqX_ddf(GEN f, GEN T) as FpX_ddf.

GEN F2xqX_degfact(GEN f, GEN T) as FpX_degfact.

7.3.34 F2xqXQ.. See FlxqXQ operations.

GEN FlxqXQ_inv(GEN x, GEN S, GEN T)

GEN FlxqXQ_invsafe(GEN x, GEN S, GEN T)

GEN F2xqXQ_mul(GEN x, GEN y, GEN S, GEN T)

GEN F2xqXQ_sqr(GEN x, GEN S, GEN T)

GEN F2xqXQ_pow(GEN x, GEN n, GEN S, GEN T)

GEN F2xqXQ_powers(GEN x, long n, GEN S, GEN T)

GEN F2xqXQ_autpow(GEN a, long n, GEN S, GEN T) as FpXQXQ_autpow

GEN F2xqXQ_auttrace(GEN a, long n, GEN S, GEN T). Let σ be the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$ and $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$; returns the vector $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \ldots + \sigma^{n-1}(b)]$ where $b = a[3]$.

GEN F2xqXQV_red(GEN x, GEN S, GEN T)

7.3.35 Functions returning objects with t_INTMOD coefficients.

Those functions are mostly needed for interface reasons: t_INTMODs should not be used in library mode since the modular kernel is more flexible and more efficient, but GP users do not have access to the modular kernel. We document them for completeness:

GEN Fp_to_mod(GEN z, GEN p), z a t_INT. Returns z * Mod(1,p), normalized. Hence the returned value is a t_INTMOD.

GEN FpX_to_mod(GEN z, GEN p), z a ZX. Returns z * Mod(1,p), normalized. Hence the returned value has t_INTMOD coefficients.

GEN FpC_to_mod(GEN z, GEN p), z a ZC. Returns Col(z) * Mod(1,p), a t_COL with t_INTMOD coefficients.

GEN FpV_to_mod(GEN z, GEN p), z a ZV. Returns Vec(z) * Mod(1,p), a t_VEC with t_INTMOD coefficients.

GEN FpVV_to_mod(GEN z, GEN p), z a ZVV. Returns Vec(z) * Mod(1,p), a t_VEC of t_VEC with t_INTMOD coefficients.

GEN FpM_to_mod(GEN z, GEN p), z a ZM. Returns z * Mod(1,p), with t_INTMOD coefficients.

GEN F2c_to_mod(GEN x)

GEN F3c_to_mod(GEN x)

GEN F2m_to_mod(GEN x)

GEN F3m_to_mod(GEN x)

GEN Flc_to_mod(GEN z)

GEN Flm_to_mod(GEN z)

GEN FqC_to_mod(GEN z, GEN T, GEN p)

GEN FqM_to_mod(GEN z, GEN T, GEN p)

GEN FpXC_to_mod(GEN V, GEN p)

GEN FpXM_to_mod(GEN V, GEN p)

GEN FpXQC_to_mod(GEN V, GEN T, GEN p) V being a vector of FpXQ, converts each entry to a t_POLMOD with t_INTMOD coefficients, and return a t_COL.

GEN FpXQX_to_mod(GEN P, GEN T, GEN p) P being a FpXQX, converts each coefficient to a t_POLMOD with t_INTMOD coefficients.

GEN FqX_to_mod(GEN P, GEN T, GEN p) same but allow T = NULL.

GEN FqXC_to_mod(GEN P, GEN T, GEN p)

GEN FqXM_to_mod(GEN P, GEN T, GEN p)

GEN QXQ_to_mod_shallow(GEN x, GEN T) x a QXQ, which is a lifted representative of elements of $\mathbf{Q}[X]/(T)$ (number field elements in most applications) and T is in $\mathbf{Z}[X]$. Convert it to a t_POLMOD modulo T; no reduction mod T is attempted: the representatives should be already reduced. Shallow function.

GEN QXQV_to_mod(GEN V, GEN T) V a vector of QXQ, which are lifted representatives of elements of $\mathbf{Q}[X]/(T)$ (number field elements in most applications) and T is in $\mathbf{Z}[X]$. Return a vector where all nonrational entries are converted to t_POLMOD modulo T; no reduction mod T is attempted: the representatives should be already reduced. Used to normalize the output of nfroots.

GEN QXQX_to_mod_shallow(GEN P, GEN T) P a polynomial with QXQ coefficients; replace them by mkpolmod(.,T). Shallow function.

GEN QXQC_to_mod_shallow(GEN V, GEN T) V a vector with QXQ coefficients; replace them by mkpolmod(.,T). Shallow function.

GEN QXQM_to_mod_shallow(GEN M, GEN T) M a matrix with QXQ coefficients; replace them by mkpolmod(.,T). Shallow function.

GEN QXQXV_to_mod(GEN V, GEN T) V a vector of polynomials whose coefficients are QXQ. Analogous to QXQV_to_mod. Used to normalize the output of nffactor.

The following functions are obsolete and should not be used: they receive a polynomial with arbitrary coefficients, apply a conversion function to map them to a finite field, a function from the modular kernel, then *_to_mod:

GEN rootmod(GEN f, GEN p), applies FpX_roots.

GEN rootmod2(GEN f, GEN p), (now) identical to rootmod.

GEN rootmod0(GEN f, GEN p, long flag), (now) identical to rootmod; ignores *flag*.

GEN factmod(GEN f, GEN p) applies *_factor.

GEN simplefactmod(GEN f, GEN p) applies *_degfact.

7.3.36 Slow Chinese remainder theorem over Z. The routines in this section have quadratic time complexity with respect to the input size; see the routines in the next two sections for quasi-linear time variants.

GEN Z_chinese(GEN a, GEN b, GEN A, GEN B) returns the integer in $[0, \mathrm{lcm}(A,B)[$ congruent to a mod A and b mod B, assuming it exists; in other words, that a and b are congruent mod $\gcd(A,B)$.

GEN Z_chinese_all(GEN a, GEN b, GEN A, GEN B, GEN *pC) as Z_chinese, setting *pC to the lcm of A and B.

GEN Z_chinese_coprime(GEN a, GEN b, GEN A, GEN B, GEN C), as Z_chinese, assuming that $\gcd(A,B) = 1$ and that $C = \mathrm{lcm}(A,B) = AB$.

ulong u_chinese_coprime(ulong a, ulong b, ulong A, ulong B, ulong C), as Z_chinese_coprime for ulong inputs and output.

void Z_chinese_pre(GEN A, GEN B, GEN *pC, GEN *pU, GEN *pd) initializes chinese remainder computations modulo A and B. Sets *pC to $\mathrm{lcm}(A,B)$, *pd to $\gcd(A,B)$, *pU to an integer congruent to 0 mod (A/d) and 1 mod (B/d). It is allowed to set pd = NULL, in which case, d is still computed, but not saved.

GEN Z_chinese_post(GEN a, GEN b, GEN C, GEN U, GEN d) returns the solution to the chinese remainder problem x congruent to a mod A and b mod B, where C, U, d were set in Z_chinese_pre. If d is NULL, assume the problem has a solution. Otherwise, return NULL if it has no solution.

The following pair of functions is used in homomorphic imaging schemes, when reconstructing an integer from its images modulo pairwise coprime integers. The idea is as follows: we want to discover an integer H which satisfies $|H| < B$ for some known bound B; we are given pairs (H_p, p) with H congruent to H_p mod p and all p pairwise coprime.

Given `H` congruent to H_p modulo a number of p, whose product is q, and a new pair $(\texttt{Hp}, \texttt{p})$, `p` coprime to q, the following incremental functions use the chinese remainder theorem (CRT) to find a new `H`, congruent to the preceding one modulo q, but also to `Hp` modulo `p`. It is defined uniquely modulo qp, and we choose the centered representative. When P is larger than $2B$, we have $\texttt{H} = H$, but of course, the value of `H` may stabilize sooner. In many applications it is possible to directly check that such a partial result is correct.

`GEN Z_init_CRT(ulong Hp, ulong p)` given a `Fl` `Hp` in $[0, p-1]$, returns the centered representative `H` congruent to `Hp` modulo `p`.

`int Z_incremental_CRT(GEN *H, ulong Hp, GEN *q, ulong p)` given a `t_INT` `*H`, centered modulo `*q`, a new pair $(\texttt{Hp}, \texttt{p})$ with `p` coprime to `q`, this function updates `*H` so that it also becomes congruent to $(\texttt{Hp}, \texttt{p})$, and `*q` to the product$\texttt{qp} = \texttt{p} \cdot *\texttt{q}$. It returns 1 if the new value is equal to the old one, and 0 otherwise.

`GEN chinese1_coprime_Z(GEN v)` an alternative divide-and-conquer implementation: v is a vector of `t_INTMOD` with pairwise coprime moduli. Return the `t_INTMOD` solving the corresponding chinese remainder problem. This is a streamlined version of

`GEN chinese1(GEN v)`, which solves a general chinese remainder problem (not necessarily over **Z**, moduli not assumed coprime).

As above, for H a `ZM`: we assume that H and all `Hp` have dimension > 0. The original `*H` is destroyed.

`GEN ZM_init_CRT(GEN Hp, ulong p)`

`int ZM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above for H a `ZX`: note that the degree may increase or decrease. The original `*H` is destroyed.

`GEN ZX_init_CRT(GEN Hp, ulong p, long v)`

`int ZX_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above, for H a matrix whose coefficient are `ZX`. The original `*H` is destroyed. The entries of H are not normalized, use `ZX_renormalize` for this.

`GEN ZXM_init_CRT(GEN Hp, long deg, ulong p)` where `deg` is the maximal degree of all the `Hp`

`int ZXM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

7.3.37 Fast remainders.

The routines in these section are asymptotically fast (quasi-linear time in the input size).

`GEN Z_ZV_mod(GEN A, GEN P)` given a `t_INT` A and a vector P of positive pairwise coprime integers of length $n \geq 1$, return a vector B of the same length such that $B[i] = A \pmod{P[i]}$ and $0 \leq B[i] < P[i]$ for all $1 \leq i \leq n$. The vector P may be a `t_VEC` or a `t_VECSMALL` (treated as `ulong`s) and B has the same type as P.

`GEN Z_nv_mod(GEN A, GEN P)` given a `t_INT` A and a `t_VECSMALL` P of positive pairwise coprime integers of length $n \geq 1$, return a `t_VECSMALL` B of the same length such that $B[i] = A \pmod{P[i]}$ and $0 \leq B[i] < P[i]$ for all $1 \leq i \leq n$. The entries of P and B are treated as `ulong`s.

The following low level functions allow precomputations:

`GEN ZV_producttree(GEN P)` where P is a vector of integers (or `t_VECSMALL`) of length $n \geq 1$, return the vector of `t_VEC`s $[f(P), f^2(P), \ldots, f^k(P)]$ where f is the transformation $[p_1, p_2, \ldots, p_m] \mapsto [p_1p_2, p_3p_4, \ldots, p_{m-1}p_m]$ if m is even and $[p_1p_2, p_3p_4, \ldots, p_{m-2}p_{m-1}, p_m]$ if m is odd, and $k = O(\log m)$ is minimal so that $f^k(P)$ has length 1; in other words, $f^k(P) = [p_1p_2 \ldots p_m]$.

`GEN Z_ZV_mod_tree(GEN A, GEN P, GEN T)` as `Z_ZV_mod` where T is the tree `ZV_producttree(P)`.

`GEN ZV_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZV` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flv` $[A \pmod{P[1]}, \ldots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

`GEN ZM_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZM` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flm` $[A \pmod{P[1]}, \ldots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

`GEN ZX_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZX` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flx` polynomials $[A \pmod{P[1]}, \ldots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

`GEN ZXC_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZXC` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxC` $[A \pmod{P[1]}, \ldots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

`GEN ZXM_nv_mod_tree(GEN A, GEN P, GEN T)` A being a `ZXM` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxM` $[A \pmod{P[1]}, \ldots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

`GEN ZXX_nv_mod_tree(GEN A, GEN P, GEN T, long v)` A being a `ZXX`, and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxX` $[A \pmod{P[1]}, \ldots, A \pmod{P[n]}]$, where T is assumed to be the tree created by `ZV_producttree(P)`.

7.3.38 Fast Chinese remainder theorem over Z. The routines in these section are asymptotically fast (quasi-linear time in the input size) and should be used whenever the moduli are known from the start.

The simplest function is

`GEN ZV_chinese(GEN A, GEN P, GEN *pM)` let P be a vector of positive pairwise coprime integers, let A be a vector of integers of the same length $n \geq 1$ such that $0 \leq A[i] < P[i]$ for all i, and let M be the product of the elements of P. Returns the integer in $[0, M[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pM` is not `NULL`, set `*pM` to M. We also allow `t_VECSMALL`s for A and P (seen as vectors of unsigned integers).

`GEN ZV_chinese_center(GEN A, GEN P, GEN *pM)` As `ZV_chinese` but return integers in $[-M/2, M/2[$ instead.

The following functions allow to solve many Chinese remainder problems simultaneously, for a given set of moduli:

`GEN nxV_chinese_center(GEN A, GEN P, GEN *pt_mod)` where A is a vector of `nx` and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the `t_POL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pt_mod` is not `NULL`, set `*pt_mod` to M.

`GEN ncV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `VECSMALL`s (seen as vectors of unsigned integers) and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the `t_COL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pM` is not `NULL`, set `*pt_mod` to M.

`GEN nmV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `MATSMALL`s (seen as matrices of unsigned integers) and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the matrix whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pM` is not `NULL`, set `*pM` to M. N.B.: this function uses the parallel GP interface.

`GEN nxCV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `nxC`s and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the `t_COL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pM` is not `NULL`, set `*pt_mod` to M.

`GEN nxMV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `nxM`s and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P, returns the matrix whose entries are integers in $[-M/2, M/2[$ congruent to $A[i]$ mod $P[i]$ for all $1 \leq i \leq n$. If `pM` is not `NULL`, set `*pM` to M. N.B.: this function uses the parallel GP interface.

The other routines allow for various precomputations :

`GEN ZV_chinesetree(GEN P, GEN T)` given P a vector of integers (or `t_VECSMALL`) and a product tree T from `ZV_producttree`(P) for the same P, return a "chinese remainder tree" R, preconditionning the solution of Chinese remainder problems modulo the $P[i]$.

`GEN ZV_chinese_tree(GEN A, GEN P, GEN T, GEN R)` return `ZV_chinese`$(A, P,$ `NULL`$)$, where T is created by `ZV_producttree`(P) and R by `ZV_chinesetree`(P, T).

GEN ncV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R) as ncV_chinese_center where T is assumed to be the tree created by ZV_producttree(P) and R by ZV_chinesetree(P, T).

GEN nmV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R) as nmV_chinese_center where T is assumed to be the tree created by ZV_producttree(P) and R by ZV_chinesetree(P, T).

GEN nxV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R) as nxV_chinese_center where T is assumed to be the tree created by ZV_producttree(P) and R by ZV_chinesetree(P, T).

GEN nxCV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R) as nxCV_chinese_center where T is assumed to be the tree created by ZV_producttree(P) and R by ZV_chinesetree(P, T).

7.3.39 Rational reconstruction.

int Fp_ratlift(GEN x, GEN m, GEN amax, GEN bmax, GEN *a, GEN *b). Assuming that $0 \leq x < m$, amax ≥ 0, and bmax > 0 are t_INTs, and that 2amaxbmax $< m$, attempts to recognize x as a rational a/b, i.e. to find t_INTs a and b such that

- $a \equiv bx$ modulo m,
- $|a| \leq$ amax, $0 < b \leq$ bmax,
- $\gcd(m, b) = \gcd(a, b)$.

If unsuccessful, the routine returns 0 and leaves a, b unchanged; otherwise it returns 1 and sets a and b.

In almost all applications, we actually know that a solution exists, as well as a nonzero multiple B of b, and $m = p^\ell$ is a prime power, for a prime p chosen coprime to B hence to b. Under the single assumption $\gcd(m, b) = 1$, if a solution a, b exists satisfying the three conditions above, then it is unique.

GEN FpM_ratlift(GEN M, GEN m, GEN amax, GEN bmax, GEN denom) given an FpM modulo m with reduced or Fp_center-ed entries, reconstructs a matrix with rational coefficients by applying Fp_ratlift to all entries. Assume that all preconditions for Fp_ratlift are satisfied, as well $\gcd(m, b) = 1$ (so that the solution is unique if it exists). Return NULL if the reconstruction fails, and the rational matrix otherwise. If denom is not NULL check further that all denominators divide denom.

The function is not stack clean if one of the coefficients of M is negative (centered residues), but still suitable for gerepileupto.

GEN FpX_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom) as FpM_ratlift, where P is an FpX.

GEN FpC_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom) as FpM_ratlift, where P is an FpC.

7.3.40 Zp.

`GEN Zp_invlift(GEN b, GEN a, GEN p, long e)` let p be a prime `t_INT`, a be a `t_INT` and b a `t_INT` such that $ab \equiv 1 \bmod p$. Returns an `t_INT` A such that $A \equiv a \pmod p$ and $Ab \equiv 1 \bmod p^e$.

`GEN Zp_inv(GEN b, GEN p, long e)` let p be a prime `t_INT` and b be a `t_INT` Returns an `t_INT` A such that $Ab \equiv 1 \bmod p^e$.

`GEN Zp_div(GEN a, GEN b, GEN p, long e)` let p be a prime `t_INT` and a and b be a `t_INT` Returns an `t_INT` c such that $cb \equiv a \bmod p^e$.

`GEN Zp_sqrt(GEN b, GEN p, long e)` b and p being `t_INT`s, with p a prime (possibly 2), returns a `t_INT` a such that $a^2 \equiv b \bmod p^e$.

`GEN Z2_sqrt(GEN b, long e)` b being a `t_INT`s returns a `t_INT` a such that $a^2 \equiv b \bmod 2^e$.

`GEN Zp_sqrtlift(GEN b, GEN a, GEN p, long e)` let a, b, p be `t_INT`s, with $p > 2$, such that $a^2 \equiv b \bmod p$. Returns a `t_INT` A such that $A^2 \equiv b \bmod p^e$. Special case of `Zp_sqrtnlift`.

`GEN Zp_sqrtnlift(GEN b, GEN n, GEN a, GEN p, long e)` let a, b, n, p be `t_INT`s, with $n, p > 1$, and p coprime to n, such that $a^n \equiv b \bmod p$. Returns a `t_INT` A such that $A^n \equiv b \bmod p^e$. Special case of `ZpX_liftroot`.

`GEN Zp_teichmuller(GEN x, GEN p, long e, GEN pe)` for p an odd prime, x a `t_INT` coprime to p, and $pe = p^e$, returns the $(p-1)$-th root of 1 congruent to x modulo p, modulo p^e. For convenience, $p = 2$ is also allowed and we return 1 (x is 1 mod 4) or $2^e - 1$ (x is 3 mod 4).

`GEN teichmullerinit(long p, long n)` returns the values of `Zp_teichmuller` at all $x = 1, \ldots, p-1$.

`GEN Zp_exp(GEN z, GEN p, ulong e)` given a `t_INT` z (preferably reduced mod p^e), return $\exp_p(a)$ mod p^e (`t_INT`).

`GEN Zp_log(GEN z, GEN p, ulong e)` given a `t_INT` z (preferably reduced mod p^e), such that $a \equiv 1 \pmod p$, return $\log_p(a)$ mod p^e (`t_INT`).

7.3.41 ZpM.

`GEN ZpM_invlift(GEN M, GEN Np, GEN p, long e)` let p be a prime `t_INT`, Np be a `FpM` (modulo p) and M a `ZpM` such that $MNp \equiv 1 \bmod p$. Returns an `ZpM` N such that $N \equiv Np \pmod p$ and $MN \equiv 1 \bmod p^e$.

7.3.42 ZpX.

`GEN ZpX_roots(GEN f, GEN p, long e)` f a `ZX` with leading term prime to p, and without multiple roots mod p. Return a vector of `t_INT`s which are the roots of f mod p^e.

`GEN ZpX_liftroot(GEN f, GEN a, GEN p, long e)` f a `ZX` with leading term prime to p, and a a root mod p such that $v_p(f'(a)) = 0$. Return a `t_INT` which is the root of f mod p^e congruent to a mod p.

`GEN ZX_Zp_root(GEN f, GEN a, GEN p, long e)` same as `ZpX_liftroot` without the assumption $v_p(f'(a)) = 0$. Return a `t_VEC` of `t_INT`s, which are the p-adic roots of f congruent to a mod p (given modulo p^e). Assume that $0 \le a < p$.

`GEN ZpX_liftroots(GEN f, GEN S, GEN p, long e)` f a `ZX` with leading term prime to p, and S a vector of simple roots mod p. Return a vector of `t_INT`s which are the root of f mod p^e congruent to the $S[i]$ mod p.

GEN ZpX_liftfact(GEN A, GEN B, GEN pe, GEN p, long e) is the routine underlying polhensellift. Here, p is prime defines a finite field $\mathbf{F}_p$. A is a polynomial in $\mathbf{Z}[X]$, whose leading coefficient is nonzero in $\mathbf{F}_q$. B is a vector of monic FpX, pairwise coprime in $\mathbf{F}_p[X]$, whose product is congruent to $A/\mathrm{lc}(A)$ in $\mathbf{F}_p[X]$. Lifts the elements of B mod $\mathtt{pe} = p^e$.

GEN ZpX_Frobenius(GEN T, GEN p, ulong e) returns the p-adic lift of the Frobenius automorphism of $\mathbf{F}_p[X]/(T)$ to precision e.

long ZpX_disc_val(GEN f, GEN p) returns the valuation at p of the discriminant of f. Assume that f is a monic *separable* ZX and that p is a prime number. Proceeds by dynamically increasing the p-adic accuracy; infinite loop if the discriminant of f is 0.

long ZpX_resultant_val(GEN f, GEN g, GEN p, long M) returns the valuation at p of $\mathrm{Res}(f, g)$. Assume f, g are both ZX, and that p is a prime number coprime to the leading coefficient of f. Proceeds by dynamically increasing the p-adic accuracy. To avoid an infinite loop when the resultant is 0, we return M if the Sylvester matrix mod p^M still does not have maximal rank.

GEN ZpX_gcd(GEN f, GEN g, GEN p, GEN pm) f a monic ZX, g a ZX, $\mathtt{pm} = p^m$ a prime power. There is a unique integer $r \geq 0$ and a monic $h \in \mathbf{Q}_p[X]$ such that

$$p^r h\mathbf{Z}_p[X] + p^m\mathbf{Z}_p[X] = f\mathbf{Z}_p[X] + g\mathbf{Z}_p[X] + p^m\mathbf{Z}_p[X].$$

Return the 0 polynomial if $r \geq m$ and a monic $h \in \mathbf{Z}[1/p][X]$ otherwise (whose valuation at p is $> -m$).

GEN ZpX_reduced_resultant(GEN f, GEN g, GEN p, GEN pm) f a monic ZX, g a ZX, $\mathtt{pm} = p^m$ a prime power. The p-adic *reduced resultant* of f and g is 0 if f, g not coprime in $\mathbf{Z}_p[X]$, and otherwise the generator of the form p^d of

$$(f\mathbf{Z}_p[X] + g\mathbf{Z}_p[X]) \cap \mathbf{Z}_p.$$

Return the reduced resultant modulo p^m.

GEN ZpX_reduced_resultant_fast(GEN f, GEN g, GEN p, long M) f a monic ZX, g a ZX, p a prime. Returns the p-adic reduced resultant of f and g modulo p^M. This function computes resultants for a sequence of increasing p-adic accuracies (up to M p-adic digits), returning as soon as it obtains a nonzero result. It is very inefficient when the resultant is 0, but otherwise usually more efficient than computations using a priori bounds.

GEN ZpX_monic_factor(GEN f, GEN p, long M) f a monic ZX, p a prime, return the p-adic factorization of f, modulo p^M. This is the underlying low-level recursive function behind factorpadic (using a combination of Round 4 factorization and Hensel lifting); the factors are not sorted and the function is not gerepile-clean.

GEN ZpX_primedec(GEN T, GEN p) T a monic separable ZX, p a prime, return as a factorization matrix the shape of the prime ideal decomposition of (p) in $\mathbf{Q}[X]/(T)$: the first column contains inertia degrees, the second columns contains ramification degrees.

7.3.43 ZpXQ.

GEN ZpXQ_invlift(GEN b, GEN a, GEN T, GEN p, long e) let p be a prime t_INT, a be a FpXQ (modulo (p,T)) and b a ZpXQ such that $ab \equiv 1 \bmod (p,T)$. Returns an ZpXQ A such that $A \equiv a \pmod{p}$ and $Ab \equiv 1 \bmod (p^e,T)$.

GEN ZpXQ_inv(GEN b, GEN T, GEN p, long e) let p be a prime t_INT and b be a FpXQ (modulo T,p^e). Returns an FpXQ A such that $Ab \equiv 1 \bmod (p^e,T)$.

GEN ZpXQ_div(GEN a, GEN b, GEN T, GEN q, GEN p, long e) let p be a prime t_INT and a and b be a FpXQ (modulo T,p^e). Returns an FpXQ c such that $cb \equiv a \bmod (p^e,T)$. The parameter q must be equal to p^e.

GEN ZpXQ_sqrtnlift(GEN b, GEN n, GEN a, GEN T, GEN p, long e) let n,p be t_INTs, with $n,p>1$ and p coprime to n, and a,b be FpXQs (modulo T) such that $a^n \equiv b \bmod (p,T)$. Returns an Fq A such that $A^n \equiv b \bmod (p^e,T)$.

GEN ZpXQ_sqrt(GEN b, GEN T, GEN p, long e) let p being a odd prime and b be a FpXQ (modulo T,p^e), returns a such that $a^2 \equiv b \bmod (p^e,T)$.

GEN ZpX_ZpXQ_liftroot(GEN f, GEN a, GEN T, GEN p, long e) as ZpXQX_liftroot, but f is a polynomial in $\mathbf{Z}[X]$.

GEN ZpX_ZpXQ_liftroot_ea(GEN f, GEN a, GEN T, GEN p, long e, void *E, GEN early(void *E, GEN x, GEN q)) as ZpX_ZpXQ_liftroot with early abort: the function early(E,x,q) will be called with x is a root of f modulo $q=p^n$ for some n. If early returns a non-NULL value z, the function returns z immediately.

GEN ZpXQ_log(GEN a, GEN T, GEN p, long e) T being a ZpX irreducible modulo p, return the logarithm of a in $\mathbf{Z}_p[X]/(T)$ to precision e, assuming that $a \equiv 1 \pmod{p\mathbf{Z}_p[X]}$ if p odd or $a \equiv 1 \pmod{4\mathbf{Z}_2[X]}$ if $p=2$.

7.3.44 Zq.

GEN Zq_sqrtnlift(GEN b, GEN n, GEN a, GEN T, GEN p, long e)

7.3.45 ZpXQM.

GEN ZpXQM_prodFrobenius(GEN M, GEN T, GEN p, long e) returns the product of matrices $M\sigma(M)\sigma^2(M)\ldots\sigma^{n-1}(M)$ to precision e where σ is the lift of the Frobenius automorphism over $\mathbf{Z}_p[X]/(T)$ and n is the degree of T.

7.3.46 ZpXQX.

GEN ZpXQX_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e) is the routine underlying polhensellift. Here, p is prime, $T(Y)$ defines a finite field $\mathbf{F}_q$. A is a polynomial in $\mathbf{Z}[X,Y]$, whose leading coefficient is nonzero in $\mathbf{F}_q$. B is a vector of monic or FqX, pairwise coprime in $\mathbf{F}_q[X]$, whose product is congruent to $A/\mathrm{lc}(A)$ in $\mathbf{F}_q[X]$. Lifts the elements of B mod pe $=p^e$, such that the congruence now holds mod (T,p^e).

GEN ZpXQX_liftroot(GEN f, GEN a, GEN T, GEN p, long e) as ZpX_liftroot, but f is now a polynomial in $\mathbf{Z}[X,Y]$ and lift the root a in the unramified extension of $\mathbf{Q}_p$ with residue field $\mathbf{F}_p[Y]/(T)$, assuming $v_p(f(a))>0$ and $v_p(f'(a))=0$.

GEN ZpXQX_liftroot_vald(GEN f, GEN a, long v, GEN T, GEN p, long e) returns the foots of f as ZpXQX_liftroot, where v is the valuation of the content of f' and it is required that $v_p(f(a)) > v$ and $v_p(f'(a)) = v$.

GEN ZpXQX_roots(GEN F, GEN T, GEN p, long e)

GEN ZpXQX_liftroots(GEN F, GEN S, GEN T, GEN p, long e)

GEN ZpXQX_divrem(GEN x, GEN Sp, GEN T, GEN q, GEN p, long e, GEN *pr) as FpXQX_divrem. The parameter q must be equal to p^e.

GEN ZpXQX_digits(GEN x, GEN B, GEN T, GEN q, GEN p, long e) As FpXQX_digits. The parameter q must be equal to p^e.

GEN ZpXQX_ZpXQXQ_liftroot(GEN f, GEN a, GEN S, GEN T, GEN p, long e) as ZpXQX_liftroot, except that a is an element of $\mathbf{Z}_p[X,Y]/(S(X,Y),T(X))$.

7.3.47 ZqX. ZqX are either ZpX or ZpXQX depending whether T is NULL or not.

GEN ZqX_roots(GEN F, GEN T, GEN p, long e)

GEN ZqX_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e)

GEN ZqX_liftroot(GEN f, GEN a, GEN T, GEN p, long e)

GEN ZqX_ZqXQ_liftroot(GEN f, GEN a, GEN P, GEN T, GEN p, long e)

7.3.48 Other p-adic functions.

GEN ZpM_echelon(GEN M, long early_abort, GEN p, GEN pm) given a ZM M, a prime p and pm $= p^m$, returns an echelon form E for M mod p^m. I.e. there exist a square integral matrix U with $\det U$ coprime to p such that $E = MU$ modulo p^m. I early_abort is nonzero, return NULL as soon as one pivot in the echelon form is divisible by p^m. The echelon form is an upper triangular HNF, we do not waste time to reduce it to Gauss-Jordan form.

GEN zlm_echelon(GEN M, long early_abort, ulong p, ulong pm) variant of ZpM_echelon, for a Zlm M.

GEN ZlM_gauss(GEN a, GEN b, ulong p, long e, GEN C) as gauss with the following peculiarities: a and b are ZM, such that a is invertible modulo p. Optional C is an Flm that is an inverse of $a \bmod p$ or NULL. Return the matrix x such that $ax = b \bmod p^e$ and all elements of x are in $[0, p^e - 1]$. For efficiency, it is better to reduce a and b mod p^e first.

GEN padic_to_Q(GEN x) truncate the t_PADIC to a t_INT or t_FRAC.

GEN padic_to_Q_shallow(GEN x) shallow version of padic_to_Q

GEN QpV_to_QV(GEN v) apply padic_to_Q_shallow

long padicprec(GEN x, GEN p) returns the absolute p-adic precision of the object x, by definition the minimum precision of the components of x. For a nonzero t_PADIC, this returns valp(x) + precp(x).

long padicprec_relative(GEN x) returns the relative p-adic precision of the t_INT, t_FRAC, or t_PADIC x (minimum precision of the components of x for t_POL or vector/matrices). For a t_PADIC, this returns precp(x) if $x \neq 0$, and 0 for $x = 0$.

7.3.48.1 low-level.

The following technical function returns an optimal sequence of p-adic accuracies, for a given target accuracy:

`ulong quadratic_prec_mask(long n)` we want to reach accuracy $n \geq 1$, starting from accuracy 1, using a quadratically convergent, self-correcting, algorithm; in other words, from inputs correct to accuracy l one iteration outputs a result correct to accuracy $2l$. For instance, to reach $n = 9$, we want to use accuracies $[1, 2, 3, 5, 9]$ instead of $[1, 2, 4, 8, 9]$. The idea is to essentially double the accuracy at each step, and not overshoot in the end.

Let $a_0 = 1$, $a_1 = 2, \dots, a_k = n$, be the desired sequence of accuracies. To obtain it, we work backwards and set

$$a_k = n, \quad a_{i-1} = (a_i + 1) \setminus 2.$$

This is in essence what the function returns. But we do not want to store the a_i explicitly, even as a `t_VECSMALL`, since this would leave an object on the stack. Instead, we store a_i implicitly in a bitmask `MASK`: let $a_0 = 1$, if the i-th bit of the mask is set, set $a_{i+1} = 2a_i - 1$, and $2a_i$ otherwise; in short the bits indicate the places where we do something special and do not quite double the accuracy (which would be the straightforward thing to do).

In fact, to avoid returning separately the mask and the sequence length $k + 1$, the function returns $\texttt{MASK} + 2^{k+1}$, so the highest bit of the mask indicates the length of the sequence, and the following ones give an algorithm to obtain the accuracies. This is much simpler than it sounds, here is what it looks like in practice:

```
ulong mask = quadratic_prec_mask(n);
long l = 1;
while (mask > 1) {            /* here, the result is known to accuracy l */
  l = 2*l; if (mask & 1) l--; /* new accuracy l for the iteration */
  mask >>= 1;                 /* pop low order bit */
  /* ... lift to the new accuracy ... */
}
/* we are done. At this point l = n */
```

We just pop the bits in `mask` starting from the low order bits, stop when `mask` is 1 (that last bit corresponds to the 2^{k+1} that we added to the mask proper). Note that there is nothing specific to Hensel lifts in that function: it would work equally well for an Archimedean Newton iteration.

Note that in practice, we rather use an infinite loop, and insert an

```
if (mask == 1) break;
```

in the middle of the loop: the loop body usually includes preparations for the next iterations (e.g. lifting Bezout coefficients in a quadratic Hensel lift), which are costly and useless in the *last* iteration.

7.3.49 Conversions involving single precision objects.

7.3.49.1 To single precision.

ulong Rg_to_Fl(GEN z, ulong p), z which can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a t_INT, a t_INTMOD whose modulus is divisible by p, a t_FRAC whose denominator is coprime to p, or a t_PADIC with underlying prime ℓ satisfying $p = \ell^n$ for some n (less than the accuracy of the input). Returns lift(z * Mod(1,p)), normalized, as an Fl.

ulong Rg_to_F2(GEN z), as Rg_to_Fl for $p = 2$.

ulong padic_to_Fl(GEN x, ulong p) special case of Rg_to_Fl, for a x a t_PADIC.

GEN RgX_to_F2x(GEN x), x a t_POL, returns the F2x obtained by applying Rg_to_Fl coefficientwise.

GEN RgX_to_Flx(GEN x, ulong p), x a t_POL, returns the Flx obtained by applying Rg_to_Fl coefficientwise.

GEN RgXV_to_FlxV(GEN x, ulong p), x a vector, returns the FlxV obtained by applying RgX_to_Flx coefficientwise.

GEN Rg_to_F2xq(GEN z, GEN T), z a GEN which can be mapped to $\mathbf{F}_2[X]/(T)$: anything Rg_to_Fl can be applied to, a t_POL to which RgX_to_F2x can be applied to, a t_POLMOD whose modulus is divisible by T (once mapped to a F2x), a suitable t_RFRAC. Returns z as an F2xq, normalized.

GEN Rg_to_Flxq(GEN z, GEN T, ulong p), z a GEN which can be mapped to $\mathbf{F}_p[X]/(T)$: anything Rg_to_Fl can be applied to, a t_POL to which RgX_to_Flx can be applied to, a t_POLMOD whose modulus is divisible by T (once mapped to a Flx), a suitable t_RFRAC. Returns z as an Flxq, normalized.

GEN RgX_to_FlxqX(GEN z, GEN T, ulong p), z a GEN which can be mapped to $\mathbf{F}_p[x]/(T)[X]$: anything Rg_to_Flxq can be applied to, a t_POL to which RgX_to_Flx can be applied to, a t_POLMOD whose modulus is divisible by T (once mapped to a Flx), a suitable t_RFRAC. Returns z as an FlxqX, normalized.

GEN ZX_to_Flx(GEN x, ulong p) reduce ZX x modulo p (yielding an Flx). Faster than RgX_to_Flx.

GEN ZV_to_Flv(GEN x, ulong p) reduce ZV x modulo p (yielding an Flv).

GEN ZXV_to_FlxV(GEN v, ulong p), as ZX_to_Flx, repeatedly called on the vector's coefficients.

GEN ZXT_to_FlxT(GEN v, ulong p), as ZX_to_Flx, repeatedly called on the tree leaves.

GEN ZXX_to_FlxX(GEN B, ulong p, long v), as ZX_to_Flx, repeatedly called on the polynomial's coefficients.

GEN zxX_to_FlxX(GEN z, ulong p) as zx_to_Flx, repeatedly called on the polynomial's coefficients.

GEN ZXXV_to_FlxXV(GEN V, ulong p, long v), as ZXX_to_FlxX, repeatedly called on the vector's coefficients.

GEN ZXXT_to_FlxXT(GEN V, ulong p, long v), as ZXX_to_FlxX, repeatedly called on the tree leaves.

GEN RgV_to_Flv(GEN x, ulong p) reduce the t_VEC/t_COL x modulo p, yielding a t_VECSMALL.

GEN RgM_to_Flm(GEN x, ulong p) reduce the t_MAT x modulo p.

GEN ZM_to_Flm(GEN x, ulong p) reduce ZM x modulo p (yielding an Flm).

GEN ZXC_to_FlxC(GEN x, ulong p, long sv) reduce ZXC x modulo p (yielding an FlxC). Assume that sv = evalvarn(v) where v is the variable number of the entries of x. It is allowed for the entries of x to be t_INT.

GEN ZXM_to_FlxM(GEN x, ulong p, long sv) reduce ZXM x modulo p (yielding an FlxM). Assume that sv = evalvarn(v) where v is the variable number of the entries of x. It is allowed for the entries of x to be t_INT.

GEN ZV_to_zv(GEN z), converts coefficients using itos

GEN ZV_to_nv(GEN z), converts coefficients using itou

GEN ZM_to_zm(GEN z), converts coefficients using itos

7.3.49.2 From single precision.

GEN Flx_to_ZX(GEN z), converts to ZX (t_POL of nonnegative t_INTs in this case)

GEN Flx_to_FlxX(GEN z), converts to FlxX (t_POL of constant Flx in this case).

GEN Flx_to_ZX_inplace(GEN z), same as Flx_to_ZX, in place (z is destroyed).

GEN FlxX_to_ZXX(GEN B), converts an FlxX to a polynomial with ZX or t_INT coefficients (repeated calls to Flx_to_ZX).

GEN FlxXC_to_ZXXC(GEN B), converts an FlxXC to a t_COL with ZXX coefficients (repeated calls to FlxX_to_ZXX).

GEN FlxXM_to_ZXXM(GEN B), converts an FlxXM to a t_MAT with ZXX coefficients (repeated calls to FlxX_to_ZXX).

GEN FlxC_to_ZXC(GEN x), converts a vector of Flx to a column vector of polynomials with t_INT coefficients (repeated calls to Flx_to_ZX).

GEN FlxV_to_ZXV(GEN x), as above but return a t_VEC.

void F2xV_to_FlxV_inplace(GEN v) v is destroyed.

void F2xV_to_ZXV_inplace(GEN v) v is destroyed.

void FlxV_to_ZXV_inplace(GEN v) v is destroyed.

GEN FlxM_to_ZXM(GEN z), converts a matrix of Flx to a matrix of polynomials with t_INT coefficients (repeated calls to Flx_to_ZX).

GEN zx_to_ZX(GEN z), as Flx_to_ZX, without assuming the coefficients to be nonnegative.

GEN zx_to_Flx(GEN z, ulong p) as Flx_red without assuming the coefficients to be nonnegative.

GEN Flc_to_ZC(GEN z), converts to ZC (t_COL of nonnegative t_INTs in this case)

GEN Flc_to_ZC_inplace(GEN z), same as Flc_to_ZC, in place (z is destroyed).

GEN Flv_to_ZV(GEN z), converts to ZV (t_VEC of nonnegative t_INTs in this case)

GEN Flm_to_ZM(GEN z), converts to ZM (t_MAT with nonnegative t_INTs coefficients in this case)

GEN Flm_to_ZM_inplace(GEN z), same as Flm_to_ZM, in place (z is destroyed).

GEN zc_to_ZC(GEN z) as Flc_to_ZC, without assuming coefficients are nonnegative.

GEN zv_to_ZV(GEN z) as Flv_to_ZV, without assuming coefficients are nonnegative.

GEN zm_to_ZM(GEN z) as Flm_to_ZM, without assuming coefficients are nonnegative.

GEN zv_to_Flv(GEN z, ulong p)

GEN zm_to_Flm(GEN z, ulong p)

7.3.49.3 Mixed precision linear algebra. Assumes dimensions are compatible. Multiply a multiprecision object by a single-precision one.

GEN RgM_zc_mul(GEN x, GEN y)

GEN RgMrow_zc_mul(GEN x, GEN y, long i)

GEN RgM_zm_mul(GEN x, GEN y)

GEN RgV_zc_mul(GEN x, GEN y)

GEN RgV_zm_mul(GEN x, GEN y)

GEN ZM_zc_mul(GEN x, GEN y)

GEN zv_ZM_mul(GEN x, GEN y)

GEN ZV_zc_mul(GEN x, GEN y)

GEN ZM_zm_mul(GEN x, GEN y)

GEN ZC_z_mul(GEN x, long y)

GEN ZM_nm_mul(GEN x, GEN y) the entries of y are ulongs.

GEN nm_Z_mul(GEN y, GEN c) the entries of y are ulongs.

7.3.49.4 Miscellaneous involving Fl.

GEN Fl_to_Flx(ulong x, long evx) converts a unsigned long to a scalar Flx. Assume that evx = evalvarn(vx) for some variable number vx.

GEN Z_to_Flx(GEN x, ulong p, long sv) converts a t_INT to a scalar Flx polynomial. Assume that sv = evalvarn(v) for some variable number v.

GEN Flx_to_Flv(GEN x, long n) converts from Flx to Flv with n components (assumed larger than the number of coefficients of x).

GEN zx_to_zv(GEN x, long n) as Flx_to_Flv.

GEN Flv_to_Flx(GEN x, long sv) converts from vector (coefficient array) to (normalized) polynomial in variable v.

GEN zv_to_zx(GEN x, long n) as Flv_to_Flx.

GEN Flm_to_FlxV(GEN x, long sv) converts the columns of Flm x to an array of Flx in the variable v (repeated calls to Flv_to_Flx).

GEN FlxM_to_FlxXV(GEN V, long v) see RgM_to_RgXV

GEN zm_to_zxV(GEN x, long n) as Flm_to_FlxV.

GEN Flm_to_FlxX(GEN x, long sw, long sv) same as Flm_to_FlxV(x,sv) but returns the result as a (normalized) polynomial in variable w.

GEN FlxV_to_Flm(GEN v, long n) reverse Flm_to_FlxV, to obtain an Flm with n rows (repeated calls to Flx_to_Flv).

GEN FlxX_to_Flx(GEN P) Let $P(x, X)$ be a FlxX, return $P(0, X)$ as a Flx.

GEN FlxX_to_Flm(GEN v, long n) reverse Flm_to_FlxX, to obtain an Flm with n rows (repeated calls to Flx_to_Flv).

GEN FlxX_to_FlxC(GEN B, long n, long sv) see RgX_to_RgV. The coefficients of B are assumed to be in the variable v.

GEN FlxV_to_FlxX(GEN x, long v) see RgV_to_RgX.

GEN FlxXV_to_FlxM(GEN V, long n, long sv) see RgXV_to_RgM. The coefficients of V[i] are assumed to be in the variable v.

GEN Fly_to_FlxY(GEN a, long sv) convert coefficients of a to constant Flx in variable v.

7.3.49.5 Miscellaneous involving F2x.

GEN F2x_to_F2v(GEN x, long n) converts from F2x to F2v with n components (assumed larger than the number of coefficients of x).

GEN F2xC_to_ZXC(GEN x), converts a vector of F2x to a column vector of polynomials with t_INT coefficients (repeated calls to F2x_to_ZX).

GEN F2xC_to_FlxC(GEN x)

GEN FlxC_to_F2xC(GEN x)

GEN F2xV_to_F2m(GEN v, long n) F2x_to_F2v to each polynomial to get an F2m with n rows.

7.4 Higher arithmetic over Z: primes, factorization.

7.4.1 Pure powers.

long Z_issquare(GEN n) returns 1 if the t_INT n is a square, and 0 otherwise. This is tested first modulo small prime powers, then sqrtremi is called.

long Z_issquareall(GEN n, GEN *sqrtn) as Z_issquare. If n is indeed a square, set sqrtn to its integer square root. Uses a fast congruence test mod $64 \times 63 \times 65 \times 11$ before computing an integer square root.

long Z_ispow2(GEN x) returns 1 if the t_INT x is a power of 2, and 0 otherwise.

long uissquare(ulong n) as Z_issquare, for an ulong operand n.

long uissquareall(ulong n, ulong *sqrtn) as Z_issquareall, for an ulong operand n.

ulong usqrt(ulong a) returns the floor of the square root of a.

ulong usqrtn(ulong a, ulong n) returns the floor of the n-th root of a.

long Z_ispower(GEN x, ulong k) returns 1 if the t_INT n is a k-th power, and 0 otherwise; assume that $k > 1$.

long Z_ispowerall(GEN x, ulong k, GEN *pt) as Z_ispower. If n is indeed a k-th power, set *pt to its integer k-th root.

long Z_isanypower(GEN x, GEN *ptn) returns the maximal $k \geq 2$ such that the t_INT $x = n^k$ is a perfect power, or 0 if no such k exist; in particular ispower(1), ispower(0), ispower(-1) all return 0. If the return value k is not 0 (so that $x = n^k$) and ptn is not NULL, set *ptn to n.

The following low-level functions are called by `Z_isanypower` but can be directly useful:

`int is_357_power(GEN x, GEN *ptn, ulong *pmask)` tests whether the integer $x > 0$ is a 3-rd, 5-th or 7-th power. The bits of `*mask` initially indicate which test is to be performed; bit 0: 3-rd, bit 1: 5-th, bit 2: 7-th (e.g. $*$`pmask` $= 7$ performs all tests). They are updated during the call: if the "i-th power" bit is set to 0 then x is not a k-th power. The function returns 0 (not a 3-rd, 5-th or 7-th power), 3 (3-rd power, not a 5-th or 7-th power), 5 (5-th power, not a 7-th power), or 7 (7-th power); if an i-th power bit is initially set to 0, we take it at face value and assume x is not an i-th power without performing any test. If the return value k is nonzero, set `*ptn` to n such that $x = n^k$.

`int is_pth_power(GEN x, GEN *ptn, forprime_t *T, ulong cutoff)` let $x > 0$ be an integer, `cutoff` > 0 and T be an iterator over primes ≥ 11, we look for the smallest prime p such that $x = n^p$ (advancing T as we go along). The 11 is due to the fact that `is_357_power` and `issquare` are faster than the generic version for $p < 11$.

Fail and return 0 when the existence of p would imply $2^{\texttt{cutoff}} > x^{1/p}$, meaning that a possible n is so small that it should have been found by trial division; for maximal speed, you should start by a round of trial division, but the cut-off may also be set to 1 for a rigorous result without any trial division.

Otherwise returns the smallest suitable prime power p^i and set `*ptn` to the p^i-th root of x (which is now not a p-th power). We may immediately recall the function with the same parameters after setting $x = *$`ptn`: it will start at the next prime.

7.4.2 Factorization.

`GEN Z_factor(GEN n)` factors the `t_INT n`. The "primes" in the factorization are actually strong pseudoprimes.

`GEN absZ_factor(GEN n)` returns `Z_factor(absi(n))`.

`long Z_issmooth(GEN n, ulong lim)` returns 1 if all the prime factors of the `t_INT` n are less or equal to lim.

`GEN Z_issmooth_fact(GEN n, ulong lim)` returns `NULL` if a prime factor of the `t_INT` n is $> lim$, and returns the factorization of n otherwise, as a `t_MAT` with `t_VECSMALL` columns (word-size primes and exponents). Neither memory-clean nor suitable for `gerepileupto`.

`GEN Z_factor_until(GEN n, GEN lim)` as `Z_factor`, but stop the factorization process as soon as the unfactored part is smaller than `lim`. The resulting factorization matrix only contains the factors found. No other assumptions can be made on the remaining factors.

`GEN Z_factor_limit(GEN n, ulong lim)` trial divide n by all primes $p <$ `lim` in the precomputed list of prime numbers and the `addprimes` prime table. Return the corresponding factorization matrix. The first column of the factorization matrix may contain a single composite, which may or may not be the last entry in presence of a prime table.

If `lim` $= 0$, the effect is the same as setting `lim` $=$ `maxprime()` $+ 1$: use all precomputed primes.

`GEN absZ_factor_limit(GEN n, ulong all)` returns `Z_factor_limit(absi(n))`.

`GEN absZ_factor_limit_strict(GEN n, ulong all, GEN *pU)`. This function is analogous to `absZ_factor_limit`, with a better interface: trial divide n by all primes $p <$ `lim` in the precomputed list of prime numbers and the `addprimes` prime table. Return the corresponding factorization matrix. In this case, a composite cofactor is *not* included.

If `pU` is not `NULL`, set it to the cofactor, which is either `NULL` (no cofactor) or $[q, k]$, where $k > 0$, the prime divisors of q are greater than `all`, q is not a pure power, and q^k is the largest power of q dividing n. It may happen that q is prime.

`GEN boundfact(GEN x, ulong lim)` as `Z_factor_limit`, applying to `t_INT` or `t_FRAC` inputs.

`GEN Z_smoothen(GEN n, GEN L, GEN *pP, GEN *pE)` given a `t_VEC` L containing a list of primes and a `t_INT` n, trial divide n by the elements of L and return the cofactor. Return `NULL` if the cofactor is ± 1. `*P` and `*E` contain the list of prime divisors found and their exponents, as `t_VECSMALL`s. Neither memory-clean, nor suitable for `gerepileupto`.

`GEN Z_lsmoothen(GEN n, GEN L, GEN *pP, GEN *pE)` as **Z_smoothen** where L is a `t_VECSMALL` of small primes and both `*P` and `*E` are given as `t_VECSMALL`.

`GEN Z_factor_listP(GEN N, GEN L)` given a `t_INT` N, a vector or primes L containing all prime divisors of N (and possibly others). Return `factor(N)`. Neither memory-clean, nor suitable for `gerepileupto`.

`GEN factor_pn_1(GEN p, ulong n)` returns the factorization of $p^n - 1$, where p is prime and n is a positive integer.

`GEN factor_pn_1_limit(GEN p, ulong n, ulong B)` returns a partial factorization of $p^n - 1$, where p is prime and n is a positive integer. Don't actively search for prime divisors $p > B$, but we may find still find some due to Aurifeuillian factorizations. Any entry $> B^2$ in the output factorization matrix is *a priori* not a prime (but may well be).

`GEN factor_Aurifeuille_prime(GEN p, long n)` an Aurifeuillian factor of $\phi_n(p)$, assuming p prime and an Aurifeuillian factor exists ($p\zeta_n$ is a square in $\mathbf{Q}(\zeta_n)$).

`GEN factor_Aurifeuille(GEN a, long n)` an Aurifeuillian factor of $\phi_n(a)$, assuming a is a nonzero integer and $n > 2$. Returns 1 if no Aurifeuillian factor exists.

`GEN odd_prime_divisors(GEN a)` `t_VEC` of all prime divisors of the `t_INT` a.

`GEN factoru(ulong n)`, returns the factorization of n. The result is a 2-component vector $[P, E]$, where P and E are `t_VECSMALL` containing the prime divisors of n, and the $v_p(n)$.

`GEN factoru_pow(ulong n)`, returns the factorization of n. The result is a 3-component vector $[P, E, C]$, where P, E and C are `t_VECSMALL` containing the prime divisors of n, the $v_p(n)$ and the $p^{v_p(n)}$.

`GEN vecfactoru(ulong a, ulong b)`, returns a `t_VEC` v containing the factorizations (`factoru` format) of $a, \ldots, b$; assume that $b \geq a > 0$. Uses a sieve with primes up to $\sqrt{b}$. For all c, $a \leq c \leq b$, the factorization of c is given in $v[c - a + 1]$.

`GEN vecfactoroddu(ulong a, ulong b)`, returns a `t_VEC` v containing the factorizations (`factoru` format) of odd integers in $a, \ldots, b$; assume that $b \geq a > 0$ are odd. Uses a sieve with primes up to $\sqrt{b}$. For all odd c, $a \leq c \leq b$, the factorization of c is given in in $v[(c - a)/2 + 1]$.

`GEN vecfactoru_i(ulong a, ulong b)`, private version of `vecfactoru`, not memory clean.

`GEN vecfactoroddu_i(ulong a, ulong b)`, private version of `vecfactoroddu`, not memory clean.

`GEN vecfactorsquarefreeu(ulong a, ulong b)` return a `t_VEC` v containing the prime divisors of squarefree integers in $a, \ldots, b$; assume that $a \leq b$. Uses a sieve with primes up to $\sqrt{b}$. For all squarefree c, $a \leq c \leq b$, the prime divisors of c (as a `t_VECSMALL`) are given in $v[c - a + 1]$, and the other entries are NULL. Note that because of these NULL markers, v is not a valid GEN, it is not memory clean and cannot be used in garbage collection routines.

GEN vecfactorsquarefreeu_coprime(ulong a, ulong b, GEN P) given a *sorted* t_VECSMALL of primes P, return a t_VEC v containing the prime divisors of squarefree integers in $a, \ldots, b$ coprime to the elements of P; assume that $a \leq b$. Uses a sieve with primes up to $\sqrt{b}$. For all squarefree c, $a \leq c \leq b$, the prime divisors of c (as a t_VECSMALL) are given in $v[c-a+1]$, and the other entries are NULL. Note that because of these NULL markers, v is not a valid GEN, it is not memory clean and cannot be used in garbage collection routines.

GEN vecsquarefreeu(ulong a, ulong b) return a t_VECSMALL v containing the squarefree integers in $a, \ldots, b$. Assume that $a \leq b$. Uses a sieve with primes up to $\sqrt{b}$.

ulong tridiv_bound(GEN n) returns the trial division bound used by Z_factor(n).

GEN tridiv_boundu(ulong n) returns the trial division bound used by factoru$n$.

GEN Z_pollardbrent(GEN N, long n, long seed) try to factor t_INT N using $n \geq 1$ rounds of Pollard iterations; *seed* is an integer whose value (mod 8) selects the quadratic polynomial use to generate Pollard's (pseudo)random walk. Returns NULL on failure, else a vector of 2 (possibly 3) integers whose product is N.

GEN Z_ECM(GEN N, long n, long seed, ulong B1) try to factor t_INT N using $n \geq 1$ rounds of ECM iterations (on 8 to 64 curves simultaneously, depending on the size of N); *seed* is an integer whose value selects the curves to be used: increase it by $64n$ to make sure that a subsequent call with a factor of N uses a disjoint set of curves. Finally $B_1 > 7$ determines the computations performed on the curves: we compute $[k]P$ for some point in $E(\mathbf{Z}/N\mathbf{Z})$ and $k = q\prod p^{e_p}$ where $p^{e_p} \leq B_1$ and $q \leq B_2 := 110B_1$; a higher value of B_1 means higher chances of hitting a factor and more time spent. The computation is deterministic for a given set of parameters. Returns NULL on failure, else a nontrivial factor or N.

GEN Q_factor(GEN x) as Z_factor, where x is a t_INT or a t_FRAC.

GEN Q_factor_limit(GEN x, ulong lim) as Z_factor_limit, where x is a t_INT or a t_FRAC.

7.4.3 Coprime factorization.

Given a and b two nonzero integers, let **ppi**(a,b), **ppo**(a,b), **ppg**(a,b), **pple**(a,b) (powers in a of primes inside b, outside b, greater than those in b, less than or equal to those in b) be the integers defined by

- $v_p(\text{ppi}) = v_p(a)[v_p(b) > 0]$,
- $v_p(\text{ppo}) = v_p(a)[v_p(b) = 0]$,
- $v_p(\text{ppg}) = v_p(a)[v_p(a) > v_p(b)]$,
- $v_p(\text{pple}) = v_p(a)[v_p(a) \leq v_p(b)]$.

GEN Z_ppo(GEN a, GEN b) returns $\text{ppo}(a,b)$; shallow function.

ulong u_ppo(ulong a, ulong b) returns $\text{ppo}(a,b)$.

GEN Z_ppgle(GEN a, GEN b) returns $[\text{ppg}(a,b), \text{pple}(a,b)]$; shallow function.

GEN Z_ppio(GEN a, GEN b) returns $[\gcd(a,b), \text{ppi}(a,b), \text{ppo}(a,b)]$; shallow function.

`GEN Z_cba(GEN a, GEN b)` fast natural coprime base algorithm. Returns a vector of coprime divisors of a and b such that both a and b can be multiplicatively generated from this set. Perfect powers are not removed, use `Z_isanypower` if needed; shallow function.

`GEN ZV_cba_extend(GEN P, GEN b)` extend a coprime basis P by the integer b, the result being a coprime basis for $P \cup \{b\}$. Perfect powers are not removed; shallow function.

`GEN ZV_cba(GEN v)` given a vector of nonzero integers v, return a coprime basis for v. Perfect powers are not removed; shallow function.

7.4.4 Checks attached to arithmetic functions.

Arithmetic functions accept arguments of the following kind: a plain positive integer N (`t_INT`), the factorization fa of a positive integer (a `t_MAT` with two columns containing respectively primes and exponents), or a vector $[N, fa]$. A few functions accept nonzero integers (e.g. `omega`), and some others arbitrary integers (e.g. `factorint`, ...).

`int is_Z_factorpos(GEN f)` returns 1 if f looks like the factorization of a positive integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that f is a two-column matrix all of whose entries are positive integers. It does *not* check that entries in the first column ("primes") are prime, or even pairwise coprime, nor that they are stricly increasing.

`int is_Z_factornon0(GEN f)` returns 1 if f looks like the factorization of a nonzero integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof, analogous to `is_Z_factorpos`. (Entries in the first column need only be nonzero integers.)

`int is_Z_factor(GEN f)` returns 1 if f looks like the factorization of an integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that f is a two-column matrix all of whose entries are integers. Entries in the second column ("exponents") are all positive. Either it encodes the "factorization" 0^e, $e > 0$, or entries in the first column ("primes") are all nonzero.

`GEN clean_Z_factor(GEN f)` assuming f is the factorization of an integer n, return the factorization of $|n|$, i.e. remove -1 from the factorization. Shallow function.

`GEN fuse_Z_factor(GEN f, GEN B)` assuming f is the factorization of an integer n, return `boundfact(n, B)`, i.e. return a factorization where all primary factors for $|p| \le B$ are preserved, and all others are "fused" into a single composite integer; if that remainder is trivial, i.e. equal to 1, it is of course not included. Shallow function.

In the following three routines, f is the name of an arithmetic function, and n a supplied argument. They all raise exceptions if n does not correspond to an integer or an integer factorization of the expected shape.

`GEN check_arith_pos(GEN n, const char *f)` check whether n is attached to the factorization of a positive integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise. May raise an `e_DOMAIN` ($n \le 0$) or an `e_TYPE` exception (other failures).

`GEN check_arith_non0(GEN n, const char *f)` check whether n is attached to the factorization of a nonzero integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise. May raise an `e_TYPE` exception.

`GEN check_arith_all(GEN n, const char *f)` is attached to the factorization of an integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise.

7.4.5 Incremental integer factorization.

Routines attached to the dynamic factorization of an integer n, iterating over successive prime divisors. This is useful to implement high-level routines allowed to take shortcuts given enough partial information: e.g. `moebius`(n) can be trivially computed if we hit p such that $p^2 \mid n$. For efficiency, trial division by small primes should have already taken place. In any case, the functions below assume that no prime $< 2^{14}$ divides n.

`GEN ifac_start(GEN n, int moebius)` schedules a new factorization attempt for the integer n. If `moebius` is nonzero, the factorization will be aborted as soon as a repeated factor is detected (Moebius mode). The function assumes that $n > 1$ is a *composite* `t_INT` whose prime divisors satisfy $p > 2^{14}$ *and* that one can write to n in place.

This function stores data on the stack, no `gerepile` call should delete this data until the factorization is complete. Returns `partial`, a data structure recording the partial factorization state.

`int ifac_next(GEN *partial, GEN *p, long *e)` deletes a primary factor p^e from `partial` and sets `p` (prime) and `e` (exponent), and normally returns 1. Whatever remains in the `partial` structure is now coprime to p.

Returns 0 if all primary factors have been used already, so we are done with the factorization. In this case p is set to `NULL`. If we ran in Moebius mode and the factorization was in fact aborted, we have $e = 1$, otherwise $e = 0$.

`int ifac_read(GEN part, GEN *k, long *e)` peeks at the next integer to be factored in the list k^e, where k is not necessarily prime and can be a perfect power as well, but will be factored by the next call to `ifac_next`. You can remove this factorization from the schedule by calling:

`void ifac_skip(GEN part)` removes the next scheduled factorization.

`int ifac_isprime(GEN n)` given n whose prime divisors are $> 2^{14}$, returns the decision the factoring engine would take about the compositeness of n: 0 if n is a proven composite, and 1 if we believe it to be prime; more precisely, n is a proven prime if `factor_proven` is set, and only a BPSW-pseudoprime otherwise.

7.4.6 Integer core, squarefree factorization.

`long Z_issquarefree(GEN n)` returns 1 if the `t_INT` `n` is square-free, and 0 otherwise.

`long Z_issquarefree_fact(GEN fa)` same, where `fa` is `factor(n)`.

`long Z_isfundamental(GEN x)` returns 1 if the `t_INT` `x` is a fundamental discriminant, and 0 otherwise.

`GEN core(GEN n)` unique squarefree integer d dividing n such that n/d is a square. The core of 0 is defined to be 0.

`GEN core2(GEN n)` return $[d, f]$ with d squarefree and $n = df^2$.

`GEN corepartial(GEN n, long lim)` as `core`, using `boundfact(n,lim)` to partially factor `n`. The result is not necessarily squarefree, but $p^2 \mid n$ implies $p >$ `lim`.

`GEN core2partial(GEN n, long lim)` as `core2`, using `boundfact(n,lim)` to partially factor `n`. The resulting d is not necessarily squarefree, but $p^2 \mid n$ implies $p >$ `lim`.

7.4.7 Primes, primality and compositeness tests.

7.4.7.1 Chebyshev's π function, bounds.

`ulong uprimepi(ulong n)`, returns the number of primes $p \leq n$ (Chebyshev's π function).

`double primepi_upper_bound(double x)` return a quick upper bound for $\pi(x)$, using Dusart bounds.

`GEN gprimepi_upper_bound(GEN x)` as `primepi_upper_bound`, returns a `t_REAL`.

`double primepi_lower_bound(double x)` return a quick lower bound for $\pi(x)$, using Dusart bounds.

`GEN gprimepi_lower_bound(GEN x)` as `primepi_lower_bound`, returns a `t_REAL` or `gen_0`.

7.4.7.2 Primes, primes in intervals.

`ulong unextprime(ulong n)`, returns the smallest prime $\geq n$. Return 0 if it cannot be represented as an `ulong` (n bigger than $2^{64} - 59$ or $2^{32} - 5$ depending on the word size).

`ulong uprecprime(ulong n)`, returns the largest prime $\leq n$. Return 0 if $n \leq 1$.

`ulong uprime(long n)` returns the n-th prime, assuming it fits in an `ulong` (overflow error otherwise).

`GEN prime(long n)` same as `utoi(uprime(n))`.

`GEN primes_zv(long m)` returns the first m primes, in a `t_VECSMALL`.

`GEN primes(long m)` return the first m primes, as a `t_VEC` of `t_INT`s.

`GEN primes_interval(GEN a, GEN b)` return the primes in the interval $[a, b]$, as a `t_VEC` of `t_INT`s.

`GEN primes_interval_zv(ulong a, ulong b)` return the primes in the interval $[a, b]$, as a `t_VECSMALL` of `ulongs`s.

`GEN primes_upto_zv(ulong b)` return the primes in the interval $[2, b]$, as a `t_VECSMALL` of `ulongs`s.

7.4.7.3 Tests.

`int uisprime(ulong p)`, returns 1 if `p` is a prime number and 0 otherwise.

`int uisprime_101(ulong p)`, assuming that p has no divisor ≤ 101, returns 1 if `p` is a prime number and 0 otherwise.

`int uisprime_661(ulong p)`, assuming that p has no divisor ≤ 661, returns 1 if `p` is a prime number and 0 otherwise.

`int isprime(GEN n)`, returns 1 if the `t_INT n` is a (fully proven) prime number and 0 otherwise.

`long isprimeAPRCL(GEN n)`, returns 1 if the `t_INT n` is a prime number and 0 otherwise, using only the APRCL test — not even trial division or compositeness tests. The workhorse `isprime` should be faster on average, especially if nonprimes are included!

`long isprimeECPP(GEN n)`, returns 1 if the `t_INT n` is a prime number and 0 otherwise, using only the ECPP test. The workhorse `isprime` should be faster on average.

`long BPSW_psp(GEN n)`, returns 1 if the `t_INT n` is a Baillie-Pomerance-Selfridge-Wagstaff pseudoprime, and 0 otherwise (proven composite).

`int BPSW_isprime(GEN x)` assuming x is a BPSW-pseudoprime, rigorously prove its primality. The function `isprime` is currently implemented as

```
BPSW_psp(x) && BPSW_isprime(x)
```

`long millerrabin(GEN n, long k)` performs k strong Rabin-Miller compositeness tests on the `t_INT` n, using k random bases. This function also caches square roots of -1 that are encountered during the successive tests and stops as soon as three distinct square roots have been produced; we have in principle factored n at this point, but unfortunately, there is currently no way for the factoring machinery to become aware of it. (It is highly implausible that hard to find factors would be exhibited in this way, though.) This should be slower than `BPSW_psp` for $k \geq 4$ and we expect it to be less reliable.

`GEN ecpp(GEN N)` returns an ECPP certificate for `t_INT` N; underlies `primecert`.

`GEN ecpp0(GEN N, long t)` returns a (potentially) partial ECPP certificate for `t_INT` N where strong pseudo-primes $< 2^t$ are included as primes in the certificate. Underlies `primecert` with t set to the `partial` argument.

`GEN ecppexport(GEN cert, long flag)` export a PARI ECPP certificate to MAGMA or Primo format; underlies `primecertexport`.

`long ecppisvalid(GEN cert)` checks whether a PARI ECPP certificate is valid; underlies `primecertisvalid`.

`long check_ecppcert(GEN cert)` checks whether `cert` looks like a PARI ECPP certificate, (valid or invalid) without doing any computation.

7.4.8 Iterators over primes.

`int forprime_init(forprime_t *T, GEN a, GEN b)` initialize an iterator T over primes in $[a, b]$; over primes $\geq a$ if $b =$ `NULL`. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`int forprimestep_init(forprime_t *T, GEN a, GEN b, GEN q)` initialize an iterator T over primes in an arithmetic progression in $[a, b]$; over primes $\geq a$ if $b =$ `NULL`. The argument q is either a `t_INT` ($p \equiv a \pmod{q}$) or a `t_INTMOD` `Mod(c,N)` and we restrict to that congruence class. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`GEN forprime_next(forprime_t *T)` returns the next prime in the range, assuming that T was initialized by `forprime_init`.

`int u_forprime_init(forprime_t *T, ulong a, ulong b)`

`ulong u_forprime_next(forprime_t *T)`

`void u_forprime_restrict(forprime_t *T, ulong c)` let T an iterator over primes initialized via `u_forprime_init(&T, a, b)`, possibly followed by a number of calls to `u_forprime_next`, and $a \leq c \leq b$. Restrict the range of primes considered to $[a, c]$.

`int u_forprime_arith_init(forprime_t *T, ulong a, ulong b, ulong c, ulong q)` initialize an iterator over primes in $[a, b]$, congruent to c modulo q. Subsequent calls to `u_forprime_next` will only return primes congruent to c modulo q. Note that unless $(c, q) = 1$ there will be at most one such prime.

7.5 Integral, rational and generic linear algebra.

7.5.1 ZC / ZV, ZM. A ZV (resp. a ZM, resp. a ZX) is a t_VEC or t_COL (resp. t_MAT, resp. t_POL) with t_INT coefficients.

7.5.1.1 ZC / ZV.

void RgV_check_ZV(GEN x, const char *s) Assuming x is a t_VEC or t_COL raise an error if it is not a ZV (s should point to the name of the caller).

int RgV_is_ZV(GEN x) Assuming x is a t_VEC or t_COL return 1 if it is a ZV, and 0 otherwise.

int RgV_is_ZVpos(GEN x) Assuming x is a t_VEC or t_COL return 1 if it is a ZV with positive entries, and 0 otherwise.

int RgV_is_ZVnon0(GEN x) Assuming x is a t_VEC or t_COL return 1 if it is a ZV with nonzero entries, and 0 otherwise.

int RgV_is_QV(GEN P) return 1 if the RgV P has only t_INT and t_FRAC coefficients, and 0 otherwise.

int RgV_is_arithprog(GEN v, GEN *a, GEN *b) assuming x is a t_VEC or t_COL return 1 if its entries follow an arithmetic progression of the form $a + b * n$, $n = 0, 1, \ldots$ and set a and b. Else return 0.

int ZV_equal0(GEN x) returns 1 if all entries of the ZV x are zero, and 0 otherwise.

int ZV_cmp(GEN x, GEN y) compare two ZV, which we assume have the same length (lexicographic order).

int ZV_abscmp(GEN x, GEN y) compare two ZV, which we assume have the same length (lexicographic order, comparing absolute values).

int ZV_equal(GEN x, GEN y) returns 1 if the two ZV are equal and 0 otherwise. A t_COL and a t_VEC with the same entries are declared equal.

GEN identity_ZV(long n) return the t_VEC $[1, 2, \ldots, n]$.

GEN ZC_add(GEN x, GEN y) adds x and y.

GEN ZC_sub(GEN x, GEN y) subtracts x and y.

GEN ZC_Z_add(GEN x, GEN y) adds y to x[1].

GEN ZC_Z_sub(GEN x, GEN y) subtracts y to x[1].

GEN Z_ZC_sub(GEN a, GEN x) returns the vector $[a - x_1, -x_2, \ldots, -x_n]$.

GEN ZC_copy(GEN x) returns a (t_COL) copy of x.

GEN ZC_neg(GEN x) returns −x as a t_COL.

void ZV_neg_inplace(GEN x) negates the ZV x in place, by replacing each component by its opposite (the type of x remains the same, t_COL or t_COL). If you want to save even more memory by avoiding the implicit component copies, use ZV_togglesign.

void ZV_togglesign(GEN x) negates x in place, by toggling the sign of its integer components. Universal constants gen_1, gen_m1, gen_2 and gen_m2 are handled specially and will not be corrupted. (We use togglesign_safe.)

`GEN ZC_Z_mul(GEN x, GEN y)` multiplies the `ZC` or `ZV` `x` (which can be a column or row vector) by the `t_INT` `y`, returning a `ZC`.

`GEN ZC_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZC_divexactu(GEN x, ulong y)` returns x/y assuming all divisions are exact.

`GEN ZC_Z_div(GEN x, GEN y)` returns x/y, where the resulting vector has rational entries.

`GEN ZV_ZV_mod(GEN a, GEN b)`. Assuming a and b are two `ZV` of the same length, returns the vector whose i-th component is `modii`$(a[i], b[i])$.

`GEN ZV_dotproduct(GEN x, GEN y)` as `RgV_dotproduct` assuming x and y have `t_INT` entries.

`GEN ZV_dotsquare(GEN x)` as `RgV_dotsquare` assuming x has `t_INT` entries.

`GEN ZC_lincomb(GEN u, GEN v, GEN x, GEN y)` returns $ux + vy$, where u, v are `t_INT` and x, y are `ZC` or `ZV`. Return a `ZC`

`void ZC_lincomb1_inplace(GEN X, GEN Y, GEN v)` sets $X \leftarrow X + vY$, where v is a `t_INT` and X, Y are `ZC` or `ZV`. (The result has the type of X.) Memory efficient (e.g. no-op if $v = 0$), but not gerepile-safe.

`void ZC_lincomb1_inplace_i(GEN X, GEN Y, GEN v, long n)` variant of `ZC_lincomb1_inplace`: only update $X[1], \dots, X[n]$, assuming that $n <$ `lg`(X).

`GEN ZC_ZV_mul(GEN x, GEN y, GEN p)` multiplies the `ZC` `x` (seen as a column vector) by the `ZV` `y` (seen as a row vector, assumed to have compatible dimensions).

`GEN ZV_content(GEN x)` returns the GCD of all the components of `x`.

`GEN ZV_extgcd(GEN A)` given a vector of n integers A, returns $[d, U]$, where d is the content of A and U is a matrix in $\mathrm{GL}_n(\mathbf{Z})$ such that $AU = [D, 0, \dots, 0]$.

`GEN ZV_prod(GEN x)` returns the product of all the components of `x` (1 for the empty vector).

`GEN ZV_sum(GEN x)` returns the sum of all the components of `x` (0 for the empty vector).

`long ZV_max_lg(GEN x)` returns the effective length of the longest entry in x.

`int ZV_dvd(GEN x, GEN y)` assuming x, y are two `ZV`s of the same length, return 1 if $y[i]$ divides $x[i]$ for all i and 0 otherwise. Error if one of the $y[i]$ is 0.

`GEN ZV_sort(GEN L)` sort the `ZV` L. Returns a vector with the same type as L.

`GEN ZV_sort_shallow(GEN L)` shallow version of `ZV_sort`.

`void ZV_sort_inplace(GEN L)` sort the `ZV` L, in place.

`GEN ZV_sort_uniq(GEN L)` sort the `ZV` L, removing duplicate entries. Returns a vector with the same type as L.

`GEN ZV_sort_uniq_shallow(GEN L)` shallow version of `ZV_sort_uniq`.

`long ZV_search(GEN L, GEN y)` look for the `t_INT` y in the sorted `ZV` L. Return an index i such that $L[i] = y$, and 0 otherwise.

`GEN ZV_indexsort(GEN L)` returns the permutation which, applied to the `ZV` L, would sort the vector. The result is a `t_VECSMALL`.

`GEN ZV_union_shallow(GEN x, GEN y)` given two *sorted* ZV (as per `ZV_sort`, returns the union of x and y. Shallow function. In case two entries are equal in x and y, include the one from x.

`GEN ZC_union_shallow(GEN x, GEN y)` as `ZV_union_shallow` but return a `t_COL`.

7.5.1.2 ZM.

void RgM_check_ZM(GEN A, const char *s) Assuming x is a t_MAT raise an error if it is not a ZM (s should point to the name of the caller).

GEN RgM_rescale_to_int(GEN x) given a matrix x with real entries (t_INT, t_FRAC or t_REAL), return a ZM wich is very close to Dx for some well-chosen integer D. More precisely, if the input is exact, D is the denominator of x; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

GEN ZM_copy(GEN x) returns a copy of x.

int ZM_equal(GEN A, GEN B) returns 1 if the two ZM are equal and 0 otherwise.

int ZM_equal0(GEN A) returns 1 if the ZM A is identically equal to 0.

GEN ZM_add(GEN x, GEN y) returns $\texttt{x} + \texttt{y}$ (assumed to have compatible dimensions).

GEN ZM_sub(GEN x, GEN y) returns $\texttt{x} - \texttt{y}$ (assumed to have compatible dimensions).

GEN ZM_neg(GEN x) returns $-\texttt{x}$.

void ZM_togglesign(GEN x) negates x in place, by toggling the sign of its integer components. Universal constants gen_1, gen_m1, gen_2 and gen_m2 are handled specially and will not be corrupted. (We use togglesign_safe.)

GEN ZM_mul(GEN x, GEN y) multiplies x and y (assumed to have compatible dimensions).

GEN ZM2_mul(GEN x, GEN y) multiplies the two-by-two ZM x and y.

GEN ZM_sqr(GEN x) returns x^2, where x is a square ZM.

GEN ZM_Z_mul(GEN x, GEN y) multiplies the ZM x by the t_INT y.

GEN ZM_ZC_mul(GEN x, GEN y) multiplies the ZM x by the ZC y (seen as a column vector, assumed to have compatible dimensions).

GEN ZM_ZX_mul(GEN x, GEN T) returns $x \times y$, where y is RgX_to_RgC(T, lg(x) $- 1$).

GEN ZM_diag_mul(GEN d, GEN m) given a vector d with integer entries and a ZM m of compatible dimensions, return diagonal(d) * m.

GEN ZM_mul_diag(GEN m, GEN d) given a vector d with integer entries and a ZM m of compatible dimensions, return m * diagonal(d).

GEN ZM_multosym(GEN x, GEN y)

GEN ZM_transmultosym(GEN x, GEN y)

GEN ZM_transmul(GEN x, GEN y)

GEN ZMrow_ZC_mul(GEN x, GEN y, long i) multiplies the i-th row of ZM x by the ZC y (seen as a column vector, assumed to have compatible dimensions). Assumes that x is nonempty and $0 < i < \texttt{lg}(\texttt{x}[1])$.

int ZMrow_equal0(GEN V, long i) returns 1 if the i-th row of the ZM V is zero, and 0 otherwise.

GEN ZV_ZM_mul(GEN x, GEN y) multiplies the ZV x by the ZM y. Returns a t_VEC.

GEN ZM_Z_divexact(GEN x, GEN y) returns x/y assuming all divisions are exact.

GEN ZM_divexactu(GEN x, ulong y) returns x/y assuming all divisions are exact.

GEN ZM_Z_div(GEN x, GEN y) returns x/y, where the resulting matrix has rational entries.

GEN ZM_ZV_mod(GEN a, GEN b). Assuming a is a ZM whose columns have the same length as the ZV b, apply ZV_ZV_mod($a[i], b$) to all columns.

GEN ZC_Q_mul(GEN x, GEN y) returns $x * y$, where y is a rational number and the resulting t_COL has rational entries.

GEN ZM_Q_mul(GEN x, GEN y) returns $x * y$, where y is a rational number and the resulting matrix has rational entries.

GEN ZM_pow(GEN x, GEN n) returns $\texttt{x}^{\texttt{n}}$, assuming x is a square ZM and n ≥ 0.

GEN ZM_powu(GEN x, ulong n) returns $\texttt{x}^{\texttt{n}}$, assuming x is a square ZM and n ≥ 0.

GEN ZM_det(GEN M) if M is a ZM, returns the determinant of M. This is the function underlying matdet whenever M is a ZM.

GEN ZM_permanent(GEN M) if M is a ZM, returns its permanent. This is the function underlying matpermanent whenever M is a ZM. It assumes that the matrix is square of dimension < BITS_IN_LONG.

GEN ZM_detmult(GEN M) if M is a ZM, returns a multiple of the determinant of the lattice generated by its columns. This is the function underlying detint.

GEN ZM_supnorm(GEN x) return the sup norm of the ZM x.

GEN ZM_charpoly(GEN M) returns the characteristic polynomial (in variable 0) of the ZM M.

GEN ZM_imagecompl(GEN x) returns matimagecompl(x).

long ZM_rank(GEN x) returns matrank(x).

GEN ZM_ker(GEN x) returns the primitive part of matker(x); in other words the **Q**-basis vectors are made integral and primitive.

GEN ZM_indexrank(GEN x) returns matindexrank(x).

GEN ZM_indeximage(GEN x) returns gel(ZM_indexrank(x), 2).

long ZM_max_lg(GEN x) returns the effective length of the longest entry in x.

GEN ZM_inv(GEN M, GEN *pd) if M is a ZM, return a primitive matrix H such that MH is d times the identity and set *pd to d. Uses a multimodular algorithm up to Hadamard's bound. If you suspect that the denominator is much smaller than $\det M$, you may use ZM_inv_ratlift.

GEN ZM_inv_ratlift(GEN M, GEN *pd) if M is a ZM, return a primitive matrix H such that MH is d times the identity and set *pd to d. Uses a multimodular algorithm, attempting rational reconstruction along the way. To be used when you expect that the denominator of M^{-1} is much smaller than $\det M$ else use ZM_inv.

GEN SL2_inv_shallow(GEN M) return the inverse of $M \in \mathrm{SL}_2(\mathbf{Z})$. Not gerepile-safe.

GEN ZM_pseudoinv(GEN M, GEN *pv, GEN *pd) if M is a nonempty ZM, let $v = [y, z]$ returned by indexrank and let M_1 be the corresponding square invertible matrix. Return a primitive left-inverse H such that HM_1 is d times the identity and set *pd to d. If pv is not NULL, set *pv to v. Not gerepile-safe.

GEN ZM_gauss(GEN a, GEN b) as gauss, where a and b coefficients are t_INTs.

GEN ZM_det_triangular(GEN x) returns the product of the diagonal entries of x (its determinant if it is indeed triangular).

`int ZM_isidentity(GEN x)` return 1 if the `ZM` x is the identity matrix, and 0 otherwise.

`int ZM_isdiagonal(GEN x)` return 1 if the `ZM` x is diagonal, and 0 otherwise.

`int ZM_isscalar(GEN x, GEN s)` given a `ZM` x and a `t_INT` s, return 1 if x is equal to s times the identity, and 0 otherwise. If s is `NULL`, test whether x is an arbitrary scalar matrix.

`long ZC_is_ei(GEN x)` return i if the `ZC` x has 0 entries, but for a 1 at position i.

`int ZM_ishnf(GEN x)` return 1 if x is in HNF form, i.e. is upper triangular with positive diagonal coefficients, and for $j > i$, $x_{i,i} > x_{i,j} \geq 0$.

7.5.2 QM.

`GEN QM_charpoly_ZX(GEN M)` returns the characteristic polynomial (in variable 0) of the `QM` M, assuming that the result has integer coefficients.

`GEN QM_charpoly_ZX_bound(GEN M, long b)` as `QM_charpoly_ZX` assuming that the sup norm of the (integral) result is $\leq 2^b$.

`GEN QM_gauss(GEN a, GEN b)` as `gauss`, where a and b coefficients are `t_FRAC`s.

`GEN QM_gauss_i(GEN a, GEN b, long flag)` as `QM_gauss` if `flag` is 0. Else, no longer assume that a is left-invertible and return a solution of $Pax = Pb$ where P is a row-selection matrix such that $A = PaQ$ is square invertible of maximal rank, for some column-selection matrix Q; in particular, x is a solution of the original equation $ax = b$ if and only if a solution exists.

`GEN QM_indexrank(GEN x)` returns `matindexrank(x)`.

`GEN QM_inv(GEN M)` return the inverse of the `QM` M.

`long QM_rank(GEN x)` returns `matrank(x)`.

`GEN QM_image(GEN x)` returns an integral matrix with primitive columns generating the image of x.

`GEN QM_image_shallow(GEN A)` shallow version of the previous function, not suitable for `gerepile`.

7.5.3 Qevproj.

`GEN Qevproj_init(GEN M)` let M be a $n \times d$ `ZM` of maximal rank $d \leq n$, representing the basis of a $\mathbf{Q}$-subspace V of $\mathbf{Q}^n$. Return a projector on V, to be used by `Qevproj_apply`. The interface details may change in the future, but this function currently returns $[M, B, D, p]$, where p is a `t_VECSMALL` with d entries such that the submatrix $A =$ `rowpermute`(M, p) is invertible, B is a `ZM` and d a `t_INT` such that $AB = D\mathrm{Id}_d$.

`GEN Qevproj_apply(GEN T, GEN pro)` let T be an $n \times n$ `QM`, stabilizing a $\mathbf{Q}$-subspace $V \subset \mathbf{Q}^n$ of dimension d, and let `pro` be a projector on that subspace initialized by `Qevproj_init`(M). Return the $d \times d$ matrix representing $T_{|V}$ on the basis given by the columns of M.

`GEN Qevproj_apply_vecei(GEN T, GEN pro, long k)` as `Qevproj_apply`, return only the image of the k-th basis vector $M[k]$ (still on the basis given by the columns of M).

`GEN Qevproj_down(GEN T, GEN pro)` given a `ZC` (resp. a `ZM`) T representing an element (resp. a vector of elements) in the subspace V return a `QC` (resp. a `QM`) U such that $T = MU$.

7.5.4 zv, zm.

GEN identity_zv(long n) return the t_VECSMALL $[1, 2, \ldots, n]$.

GEN random_zv(long n) returns a random zv with n components.

GEN zv_abs(GEN x) return $[|x[1]|, \ldots, |x[n]|]$ as a zv.

GEN zv_neg(GEN x) return $-x$. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\texttt{BITS\_IN\_LONG}-1}$.

GEN zv_neg_inplace(GEN x) negates x in place and return it. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\texttt{BITS\_IN\_LONG}-1}$.

GEN zm_zc_mul(GEN x, GEN y)

GEN zm_mul(GEN x, GEN y)

GEN zv_z_mul(GEN x, long n) return nx. No check for overflow is done.

long zv_content(GEN x) returns the gcd of the entries of x.

long zv_dotproduct(GEN x, GEN y)

long zv_prod(GEN x) returns the product of all the components of x (assumes no overflow occurs).

GEN zv_prod_Z(GEN x) returns the product of all the components of x; consider all $x[i]$ as ulongs.

long zv_sum(GEN x) returns the sum of all the components of x (assumes no overflow occurs).

long zv_sumpart(GEN v, long n) returns the sum $v[1] + \ldots + v[n]$ (assumes no overflow occurs and $\texttt{lg}(v) > n$).

int zv_cmp0(GEN x) returns 1 if all entries of the zv x are 0, and 0 otherwise.

int zv_equal(GEN x, GEN y) returns 1 if the two zv are equal and 0 otherwise.

int zv_equal0(GEN x) returns 1 if all entries are 0, and return 0 otherwise.

long zv_search(GEN L, long y) look for y in the sorted zv L. Return an index i such that $L[i] = y$, and 0 otherwise.

GEN zv_copy(GEN x) as Flv_copy.

GEN zm_transpose(GEN x) as Flm_transpose.

GEN zm_copy(GEN x) as Flm_copy.

GEN zero_zm(long m, long n) as zero_Flm.

GEN zero_zv(long n) as zero_Flv.

GEN zm_row(GEN A, long x0) as Flm_row.

GEN zv_diagonal(GEN v) return the square zm whose diagonal is given by the entries of v.

GEN zm_permanent(GEN M) return the permanent of M. The function assumes that the matrix is square of dimension < BITS_IN_LONG.

int zvV_equal(GEN x, GEN y) returns 1 if the two zvV (vectors of zv) are equal and 0 otherwise.

7.5.5 ZMV / zmV (vectors of ZM/zm).

int RgV_is_ZMV(GEN x) Assuming x is a t_VEC or t_COL return 1 if its components are ZM, and 0 otherwise.

GEN ZMV_to_zmV(GEN z)

GEN zmV_to_ZMV(GEN z)

GEN ZMV_to_FlmV(GEN z, ulong m)

7.5.6 QC / QV, QM.

GEN QM_mul(GEN x, GEN y) multiplies x and y (assumed to have compatible dimensions).

GEN QM_sqr(GEN x) returns the square of x (assumed to be square).

GEN QM_QC_mul(GEN x, GEN y) multiplies x and y (assumed to have compatible dimensions).

GEN QM_det(GEN M) returns the determinant of M.

GEN QM_ker(GEN x) returns matker(x).

7.5.7 RgC / RgV, RgM.

RgC and RgV routines assume the inputs are VEC or COL of the same dimension. RgM assume the inputs are MAT of compatible dimensions.

7.5.7.1 Matrix arithmetic.

void RgM_dimensions(GEN x, long *m, long *n) sets m, resp. n, to the number of rows, resp. columns of the t_MAT x.

GEN RgC_add(GEN x, GEN y) returns $x + y$ as a t_COL.

GEN RgC_neg(GEN x) returns $-x$ as a t_COL.

GEN RgC_sub(GEN x, GEN y) returns $x - y$ as a t_COL.

GEN RgV_add(GEN x, GEN y) returns $x + y$ as a t_VEC.

GEN RgV_neg(GEN x) returns $-x$ as a t_VEC.

GEN RgV_sub(GEN x, GEN y) returns $x - y$ as a t_VEC.

GEN RgM_add(GEN x, GEN y) return $x + y$.

GEN RgM_neg(GEN x) returns $-x$.

GEN RgM_sub(GEN x, GEN y) returns $x - y$.

GEN RgM_Rg_add(GEN x, GEN y) assuming x is a square matrix and y a scalar, returns the square matrix $x + y * \mathrm{Id}$.

GEN RgM_Rg_add_shallow(GEN x, GEN y) as RgM_Rg_add with much fewer copies. Not suitable for gerepileupto.

GEN RgM_Rg_sub(GEN x, GEN y) assuming x is a square matrix and y a scalar, returns the square matrix $x - y * \mathrm{Id}$.

GEN RgM_Rg_sub_shallow(GEN x, GEN y) as RgM_Rg_sub with much fewer copies. Not suitable for gerepileupto.

`GEN RgC_Rg_add(GEN x, GEN y)` assuming x is a nonempty column vector and y a scalar, returns the vector $[x_1 + y, x_2, \dots, x_n]$.

`GEN RgC_Rg_sub(GEN x, GEN y)` assuming x is a nonempty column vector and y a scalar, returns the vector $[x_1 - y, x_2, \dots, x_n]$.

`GEN Rg_RgC_sub(GEN a, GEN x)` assuming x is a nonempty column vector and a a scalar, returns the vector $[a - x_1, -x_2, \dots, -x_n]$.

`GEN RgC_Rg_div(GEN x, GEN y)`

`GEN RgM_Rg_div(GEN x, GEN y)` returns x/y (y treated as a scalar).

`GEN RgC_Rg_mul(GEN x, GEN y)`

`GEN RgV_Rg_mul(GEN x, GEN y)`

`GEN RgM_Rg_mul(GEN x, GEN y)` returns $x \times y$ (y treated as a scalar).

`GEN RgV_RgC_mul(GEN x, GEN y)` returns $x \times y$.

`GEN RgV_RgM_mul(GEN x, GEN y)` returns $x \times y$.

`GEN RgM_RgC_mul(GEN x, GEN y)` returns $x \times y$.

`GEN RgM_RgX_mul(GEN x, GEN T)` returns $x \times y$, where y is `RgX_to_RgC`$(T, \texttt{lg}(x) - 1)$.

`GEN RgM_mul(GEN x, GEN y)` returns $x \times y$.

`GEN RgM_ZM_mul(GEN x, GEN y)` returns $x \times y$ assuming that y is a `ZM`.

`GEN RgM_transmul(GEN x, GEN y)` returns $x\tilde{} \times y$.

`GEN RgM_multosym(GEN x, GEN y)` returns $x \times y$, assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).

`GEN RgM_transmultosym(GEN x, GEN y)` returns $x\tilde{} \times y$, assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).

`GEN RgMrow_RgC_mul(GEN x, GEN y, long i)` multiplies the i-th row of `RgM x` by the `RgC y` (seen as a column vector, assumed to have compatible dimensions). Assumes that x is nonempty and $0 < i < \texttt{lg}(\texttt{x}[1])$.

`GEN RgM_mulreal(GEN x, GEN y)` returns the real part of $x \times y$ (whose entries are `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`).

`GEN RgM_sqr(GEN x)` returns x^2.

`GEN RgC_RgV_mul(GEN x, GEN y)` returns $x \times y$ (the matrix $(x_i y_j)$).

`GEN RgC_RgV_mulrealsym(GEN x, GEN y)` returns the real part of $x \times y$ (whose entries are `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`), assuming the result is symmetric.

The following two functions are not well defined in general and only provided for convenience in specific cases:

`GEN RgC_RgM_mul(GEN x, GEN y)` returns $x \times y[1,]$ if y is a row matrix $1 \times n$, error otherwise.

`GEN RgM_RgV_mul(GEN x, GEN y)` returns $x \times y[,1]$ if y is a column matrix $n \times 1$, error otherwise.

`GEN RgM_powers(GEN x, long n)` returns $[\mathrm{x}^0, \dots, \mathrm{x}^\mathrm{n}]$ as a `t_VEC` of `RgM`s.

GEN RgV_sum(GEN v) sum of the entries of v

GEN RgV_prod(GEN v) product of the entries of v, using a divide and conquer strategy

GEN RgV_sumpart(GEN v, long n) returns the sum $v[1] + \ldots + v[n]$ (assumes that $\texttt{lg}(v) > n$).

GEN RgV_sumpart2(GEN v, long m, long n) returns the sum $v[m] + \ldots + v[n]$ (assumes that $\texttt{lg}(v) > n$ and $m > 0$). Returns gen_0 when $m > n$.

GEN RgM_sumcol(GEN v) returns a t_COL, sum of the columns of the t_MAT v.

GEN RgV_dotproduct(GEN x, GEN y) returns the scalar product of x and y

GEN RgV_dotsquare(GEN x) returns the scalar product of x with itself.

GEN RgV_kill0(GEN v) returns a shallow copy of v where entries matched by gequal0 are replaced by NULL. The return value is not a valid GEN and must be handled specially. The idea is to pre-treat a vector of coefficients to speed up later linear combinations or scalar products.

GEN gram_matrix(GEN v) returns the Gram matrix $(v_i \cdot v_j)$ attached to the entries of v (matrix, or vector of vectors).

GEN RgV_polint(GEN X, GEN Y, long v) X and Y being two vectors of the same length, returns the polynomial T in variable v such that $T(X[i]) = Y[i]$ for all i. The special case $X =$ NULL corresponds to $X = [1, 2, \ldots, n]$, where n is the length of Y. This is the function underlying polint for formal interpolation.

GEN polintspec(GEN X, GEN Y, GEN t, long n, long *pe) return $P(t)$ where P is the Lagrange interpolation polynomial attached to the n points $(X[0], Y[0]), \ldots, (X[n-1], Y[n-1])$. If pe is not NULL and t is a complex numeric value, *pe contains an error estimate for the returned value (Neville's algorithm, see polinterpolate). In extrapolation algorithms, e.g., Romberg integration, this function is usually called on actual GEN vectors with offsets: $x+k$ and $y+k$ so as to interpolate on $x[k..k+n-1]$ without having to use vecslice. This is the function underlying polint for numerical interpolation.

GEN polint_i(GEN X, GEN Y, GEN t, long *pe) as polintspec, where X and Y are actual GEN vectors.

GEN vandermondeinverse(GEN r, GEN T, GEN d, GEN V) Given a vector r of n scalars and the t_POL $T = \prod_{i=1}^{n} (X - r_i)$, return dM^{-1}, where $M = (r_i^{j-1})_{1 \leq i,j \leq n}$ is the van der Monde matrix; V is NULL or a vector containing the $T'(r_i)$, as returned by vandermodeinverseinit. The demonimator d may be set to NULL (handled as 1). If c is the k-column of the result, it is essentially d times the k-th Lagrange interpolation polynomial: we have $\sum_j c_j r_i^{j-1} = d\delta_{i=k}$. This is the function underlying RgV_polint when the base field is not $\mathbf{Z}/p\mathbf{Z}$: it uses $O(n^2)$ scalar operations and is asymptotically slower than variants using multi-evaluation such as FpV_polint; it is also accurate over inexact fields.

GEN vandermondeinverseinit(GEN r) Given a vector r of n scalars, let T be the t_POL $T = \prod_{j=1}^{n} (X - r_j)$. This function returns the $T'(r_i)$ computed stably via products of difference: the i-th entry is $T'(r_i) = \prod_{j \neq i} (r_i - r_j)$. It is asymptotically slow (uses $O(n^2)$ scalar operations, where multi-evaluation achieves quasi-linear running time) but allows accurate computation at low accuracies when T has large complex coefficients.

7.5.7.2 Special shapes.

The following routines check whether matrices or vectors have a special shape, using `gequal1` and `gequal0` to test components. (This makes a difference when components are inexact.)

`int RgV_isscalar(GEN x)` return 1 if all the entries of x are 0 (as per `gequal0`), except possibly the first one. The name comes from vectors expressing polynomials on the standard basis $1, T, \dots, T^{n-1}$, or on `nf.zk` (whose first element is 1).

`int QV_isscalar(GEN x)` as `RgV_isscalar`, assuming x is a `QV` (`t_INT` and `t_FRAC` entries only).

`int ZV_isscalar(GEN x)` as `RgV_isscalar`, assuming x is a `ZV` (`t_INT` entries only).

`int RgM_isscalar(GEN x, GEN s)` return 1 if x is the scalar matrix equal to s times the identity, and 0 otherwise. If s is `NULL`, test whether x is an arbitrary scalar matrix.

`int RgM_isidentity(GEN x)` return 1 if the `t_MAT` x is the identity matrix, and 0 otherwise.

`int RgM_isdiagonal(GEN x)` return 1 if the `t_MAT` x is a diagonal matrix, and 0 otherwise.

`long RgC_is_ei(GEN x)` return i if the `t_COL` x has 0 entries, but for a 1 at position i.

`int RgM_is_ZM(GEN x)` return 1 if the `t_MAT` x has only `t_INT` coefficients, and 0 otherwise.

`long qfiseven(GEN M)` return 1 if the square symmetric typZM x is an even quadratic form (all diagonal coefficients are even), and 0 otherwise.

`int RgM_is_QM(GEN x)` return 1 if the `t_MAT` x has only `t_INT` or `t_FRAC` coefficients, and 0 otherwise.

`long RgV_isin(GEN v, GEN x)` return the first index i such that $v[i] = x$ if it exists, and 0 otherwise. Naive search in linear time, does not assume that `v` is sorted.

`long RgV_isin_i(GEN v, GEN x, long n)` return the first index $i\ leqn$ such that $v[i] = x$ if it exists, and 0 otherwise. Naive search in linear time, does not assume that `v` is sorted. Assume that $n <$ `lg`(`v`).

`GEN RgM_diagonal(GEN m)` returns the diagonal of m as a `t_VEC`.

`GEN RgM_diagonal_shallow(GEN m)` shallow version of `RgM_diagonal`

7.5.7.3 Conversion to floating point entries.

`GEN RgC_gtofp(GEN x, GEN prec)` returns the `t_COL` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x.

`GEN RgV_gtofp(GEN x, GEN prec)` returns the `t_VEC` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x.

`GEN RgC_gtomp(GEN x, long prec)` returns the `t_COL` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of x.

`GEN RgC_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgC_gtofp(x, prec) )
```

`GEN RgM_gtofp(GEN x, GEN prec)` returns the `t_MAT` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x.

`GEN RgM_gtomp(GEN x, long prec)` returns the `t_MAT` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of x.

`GEN RgM_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgM_gtofp(x, prec) )
```

7.5.7.4 Linear algebra, linear systems.

`GEN RgM_inv(GEN a)` returns a left inverse of a (which needs not be square), or `NULL` if this turns out to be impossible. The latter happens when the matrix does not have maximal rank (or when rounding errors make it appear so).

`GEN RgM_inv_upper(GEN a)` as `RgM_inv`, assuming that a is a nonempty invertible upper triangular matrix, hence a little faster.

`GEN RgM_RgC_invimage(GEN A, GEN B)` returns a `t_COL` X such that $AX = B$ if one such exists, and `NULL` otherwise.

`GEN RgM_invimage(GEN A, GEN B)` returns a `t_MAT` X such that $AX = B$ if one such exists, and `NULL` otherwise.

`GEN RgM_Hadamard(GEN a)` returns a upper bound for the absolute value of $\det(a)$. The bound is a `t_INT`.

`GEN RgM_solve(GEN a, GEN b)` returns $a^{-1}b$ where a is a square `t_MAT` and b is a `t_COL` or `t_MAT`. Returns `NULL` if a^{-1} cannot be computed, see `RgM_inv`.

If $b =$ `NULL`, the matrix a need no longer be square, and we strive to return a left inverse for a (`NULL` if it does not exist).

`GEN RgM_solve_realimag(GEN M, GEN b)` M being a `t_MAT` with r_1+r_2 rows and r_1+2r_2 columns, y a `t_COL` or `t_MAT` such that the equation $Mx = y$ makes sense, returns x under the following simplifying assumptions: the first r_1 rows of M and y are real (the r_2 others are complex), and x is real. This is stabler and faster than calling `RgM_solve`(M, b) over $\mathbf{C}$. In most applications, M approximates the complex embeddings of an integer basis in a number field, and x is actually rational.

`GEN split_realimag(GEN x, long r1, long r2)` x is a `t_COL` or `t_MAT` with $r_1 + r_2$ rows, whose first r_1 rows have real entries (the r_2 others are complex). Return an object of the same type as x and $r_1 + 2r_2$ rows, such that the first $r_1 + r_2$ rows contain the real part of x, and the r_2 following ones contain the imaginary part of the last r_2 rows of x. Called by `RgM_solve_realimag`.

`GEN RgM_det_triangular(GEN x)` returns the product of the diagonal entries of x (its determinant if it is indeed triangular).

`GEN Frobeniusform(GEN V, long n)` given the vector V of elementary divisors for $M - x\mathrm{Id}$, where M is an $n \times n$ square matrix. Returns the Frobenius form of M.

`int RgM_QR_init(GEN x, GEN *pB, GEN *pQ, GEN *pL, long prec)` QR-decomposition of a square invertible `t_MAT` x with real coefficients. Sets `*pB` to the vector of squared lengths of the $x[i]$, `*pL` to the Gram-Schmidt coefficients and `*pQ` to a vector of successive Householder transforms. If R denotes the transpose of L and Q is the result of applying `*pQ` to the identity matrix, then $x = QR$ is the QR decomposition of x. Returns 0 is x is not invertible or we hit a precision problem, and 1 otherwise.

`int QR_init(GEN x, GEN *pB, GEN *pQ, GEN *pL, long prec)` as `RgM_QR_init`, assuming further that x has `t_INT` or `t_REAL` coefficients.

GEN R_from_QR(GEN x, long prec) assuming that x is a square invertible t_MAT with t_INT or t_REAL coefficients, return the upper triangular R from the QR docomposition of x. Not memory clean. If the matrix is not known to have t_INT or t_REAL coefficients, apply RgM_gtomp first.

GEN gaussred_from_QR(GEN x, long prec) assuming that x is a square invertible t_MAT with t_INT or t_REAL coefficients, returns qfgaussred(x~* x); this is essentially the upper triangular R matrix from the QR decomposition of x, renormalized to accomodate qfgaussred conventions. Not memory clean.

GEN RgM_gram_schmidt(GEN e, GEN *ptB) naive (unstable) Gram-Schmidt orthogonalization of the basis (e_i) given by the columns of t_MAT e. Return the e_i^* (as columns of a t_MAT) and set *ptB to the vector of squared lengths $|e_i^*|^2$.

GEN RgM_Babai(GEN M, GEN y) given a t_MAT M of maximal rank n and a t_COL y of the same dimension, apply Babai's nearest plane algorithm to return an *integral* x such that $y - Mx$ has small L_2 norm. This yields an approximate solution to the closest vector problem: if M is LLL-reduced, then

$$||y - Mx||_2 \leq 2(2/\sqrt{3})^n ||y - MX||_2$$

for all $X \in \mathbf{Z}^n$.

7.5.8 ZG.

Let G be a multiplicative group with neutral element 1_G whose multiplication is supported by gmul and where equality test is performed using gidentical, e.g. a matrix group. The following routines implement basic computations in the group algebra $\mathbf{Z}[G]$. All of them are shallow for efficiency reasons. A ZG is either

- a t_INT n, representing $n[1_G]$

- or a "factorization matrix" with two columns $[g, e]$: the first one contains group elements, sorted according to cmp_universal, and the second one contains integer "exponents", representing $\sum e_i[g_i]$.

Note that to_famat and to_famat_shallow(g, e) allow to build the ZG $e[g]$ from $e \in \mathbf{Z}$ and $g \in G$.

GEN ZG_normalize(GEN x) given a t_INT x or a factorization matrix *without* assuming that the first column is properly sorted. Return a valid (sorted) ZG. Shallow function.

GEN ZG_add(GEN x, GEN y) return $x + y$; shallow function.

GEN ZG_neg(GEN x) return $-x$; shallow function.

GEN ZG_sub(GEN x, GEN y) return $x - y$; shallow function.

GEN ZG_mul(GEN x, GEN y) return xy; shallow function.

GEN ZG_G_mul(GEN x, GEN y) given a ZG x and $y \in G$, return xy; shallow function.

GEN G_ZG_mul(GEN x, GEN y) given a ZG y and $x \in G$, return xy; shallow function.

GEN ZG_Z_mul(GEN x, GEN n) given a ZG x and $y \in \mathbf{Z}$, return xy; shallow function.

GEN ZGC_G_mul(GEN v, GEN x) given v a vector of ZG and $x \in G$ return the vector (with the same type as v with entries $v[i] \cdot x$. Shallow function.

void ZGC_G_mul_inplace(GEN v, GEN x) as ZGC_G_mul, modifying v in place.

`GEN ZGC_Z_mul(GEN v, GEN n)` given v a vector of `ZG` and $n \in Z$ return the vector (with the same type as v with entries $n \cdot v[i]$. Shallow function.

`GEN G_ZGC_mul(GEN x, GEN v)` given v a vector of `ZG` and $x \in G$ return the vector of $x \cdot v[i]$. Shallow function.

`GEN ZGCs_add(GEN x, GEN y)` add two sparse vectors of `ZG` elements (see Sparse linear algebra below).

7.5.9 Sparse and blackbox linear algebra.

A sparse column `zCs` v is a `t_COL` with two components C and E which are `t_VECSMALL` of the same length, representing $\sum_i E[i] * e_{C[i]}$, where (e_j) is the canonical basis. A sparse matrix (`zMs`) is a `t_VEC` of `zCs`.

`FpCs` and `FpMs` are identical to the above, but $E[i]$ is now interpreted as a *signed* C long integer representing an element of $\mathbf{F}_p$. This is important since p can be so large that $p + E[i]$ would not fit in a C long.

`RgCs` and `RgMs` are similar, except that the type of the components of E is now unspecified. Functions handling those later objects must not depend on the type of those components.

`F2Ms` are `t_VEC` of `F2Cs`. `F2Cs` are `t_VECSMALL` whoses entries are the nonzero coefficients (1).

It is not possible to derive the space dimension (number of rows) from the above data. Thus most functions take an argument `nbrow` which is the number of rows of the corresponding column/matrix in dense representation.

`GEN F2Ms_to_F2m(GEN M, long nbrow)` convert a `F2m` to a `F2Ms`.

`GEN F2m_to_F2Ms(GEN M)` convert a `F2m` to a `F2Ms`.

`GEN zCs_to_ZC(GEN C, long nbrow)` convert the sparse vector C to a dense `ZC` of dimension `nbrow`.

`GEN zMs_to_ZM(GEN M, long nbrow)` convert the sparse matrix M to a dense `ZM` whose columns have dimension `nbrow`.

`GEN FpMs_FpC_mul(GEN M, GEN B, GEN p)` multiply the sparse matrix M (over $\mathbf{F}_p$) by the `FpC` B. The result is an `FpC`, i.e. a dense vector.

`GEN zMs_ZC_mul(GEN M, GEN B, GEN p)` multiply the sparse matrix M by the `ZC` B (over $\mathbf{Z}$). The result is an `ZC`, i.e. a dense vector.

`GEN FpV_FpMs_mul(GEN B, GEN M, GEN p)` multiply the `FpV` B by the sparse matrix M (over $\mathbf{F}_p$). The result is an `FpV`, i.e. a dense vector.

`GEN ZV_zMs_mul(GEN B, GEN M, GEN p)` multiply the `FpV` B (over $\mathbf{Z}$) by the sparse matrix M. The result is an `ZV`, i.e. a dense vector.

`void RgMs_structelim(GEN M, long nbrow, GEN A, GEN *p_col, GEN *p_row)` M being a RgMs with `nbrow` rows, A being a list of row indices, perform structured elimination on M by removing some rows and columns until the number of effectively present rows is equal to the number of columns. The result is stored in two `t_VECSMALL`s, `*p_col` and `*p_row`: `*p_col` is a map from the new columns indices to the old one. `*p_row` is a map from the old rows indices to the new one (0 if removed).

`GEN F2Ms_colelim(GEN M, long nbrow)` returns some subset of the columns of M as a `t_VECSMALL` of indices, selected such that the dimension of the kernel of the matrix is preserved. The subset is not guaranteed to be minimal.

`GEN F2Ms_ker(GEN M, long nbrow)` returns some kernel vectors of M using block Lanczos algorithm.

`GEN FpMs_leftkernel_elt(GEN M, long nbrow, GEN p)` M being a sparse matrix over $\mathbf{F}_p$, return a nonzero `FpV` X such that XM components are almost all 0.

`GEN FpMs_FpCs_solve(GEN M, GEN B, long nbrow, GEN p)` solve the equation $MX = B$, where M is a sparse matrix and B is a sparse vector, both over $\mathbf{F}_p$. Return either a solution as a `t_COL` (dense vector), the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component, or `NULL` (can happen if B is not in the image of M).

`GEN FpMs_FpCs_solve_safe(GEN M, GEN B, long nbrow, GEN p)` as above, but in the event that p is not a prime and an impossible division occurs, return `NULL`.

`GEN ZpMs_ZpCs_solve(GEN M, GEN B, long nbrow, GEN p, long e)` solve the equation $MX = B$, where M is a sparse matrix and B is a sparse vector, both over $\mathbf{Z}/p^e\mathbf{Z}$. Return either a solution as a `t_COL` (dense vector), or the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component.

`GEN gen_FpM_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p)` solve the equation $f(X) = B$ over $\mathbf{F}_p$, where B is a `FpV`, and f is a blackbox endomorphism, where $f(E, X)$ computes the value of f at the (dense) column vector X. Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

`GEN gen_ZpM_Dixon_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p, long e)` solve equation $f(X) = B$ over $\mathbf{Z}/p^e\mathbf{Z}$, where B is a `ZV`, and f is a blackbox endomorphism, where $f(E, X)$ computes the value of f at the (dense) column vector X. Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

7.5.10 Obsolete functions.

The functions in this section are kept for backward compatibility only and will eventually disappear.

`GEN image2(GEN x)` compute the image of x using a very slow algorithm. Use `image` instead.

7.6 Integral, rational and generic polynomial arithmetic.

7.6.1 ZX.

`void RgX_check_ZX(GEN x, const char *s)` Assuming `x` is a `t_POL` raise an error if it is not a `ZX` (s should point to the name of the caller).

`GEN ZX_copy(GEN x, GEN p)` returns a copy of `x`.

`long ZX_max_lg(GEN x)` returns the effective length of the longest component in x.

`GEN scalar_ZX(GEN x, long v)` returns the constant `ZX` in variable v equal to the `t_INT` x.

`GEN scalar_ZX_shallow(GEN x, long v)` returns the constant `ZX` in variable v equal to the `t_INT` x. Shallow function not suitable for `gerepile` and friends.

`GEN ZX_renormalize(GEN x, long l)`, as `normalizepol`, where `l` = `lg`(x), in place.

`int ZX_equal(GEN x, GEN y)` returns 1 if the two `ZX` have the same `degpol` and their coefficients are equal. Variable numbers are not checked.

`int ZX_equal1(GEN x)` returns 1 if the `ZX` x is equal to 1 and 0 otherwise.

`int ZX_is_monic(GEN x)` returns 1 if the `ZX` x is monic and 0 otherwise. The zero polynomial considered not monic.

`GEN ZX_add(GEN x, GEN y)` adds `x` and `y`.

`GEN ZX_sub(GEN x, GEN y)` subtracts `x` and `y`.

`GEN ZX_neg(GEN x)` returns $-$x.

`GEN ZX_Z_add(GEN x, GEN y)` adds the integer `y` to the `ZX` `x`.

`GEN ZX_Z_add_shallow(GEN x, GEN y)` shallow version of `ZX_Z_add`.

`GEN ZX_Z_sub(GEN x, GEN y)` subtracts the integer `y` to the `ZX` `x`.

`GEN Z_ZX_sub(GEN x, GEN y)` subtracts the `ZX` `y` to the integer `x`.

`GEN ZX_Z_mul(GEN x, GEN y)` multiplies the `ZX` `x` by the integer `y`.

`GEN ZX_mulu(GEN x, ulong y)` multiplies `x` by the integer `y`.

`GEN ZX_shifti(GEN x, long n)` shifts all coefficients of `x` by n bits, which can be negative.

`GEN ZX_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZX_divuexact(GEN x, ulong y)` returns x/y assuming all divisions are exact.

`GEN ZX_remi2n(GEN x, long n)` reduces all coefficients of `x` to n bits, using `remi2n`.

`GEN ZX_mul(GEN x, GEN y)` multiplies `x` and `y`.

`GEN ZX_sqr(GEN x, GEN p)` returns $\texttt{x}^2$.

`GEN ZX_mulspec(GEN a, GEN b, long na, long nb)`. Internal routine: `a` and `b` are arrays of coefficients representing polynomials $\sum_{i=0}^{\texttt{na}-1} \texttt{a}[i]X^i$ and $\sum_{i=0}^{\texttt{nb}-1} \texttt{b}[i]X^i$. Returns their product (as a true `GEN`) in variable 0.

`GEN ZX_sqrspec(GEN a, long na)`. Internal routine: `a` is an array of coefficients representing polynomial $\sum_{i=0}^{\texttt{na}-1} \texttt{a}[i]X^i$. Return its square (as a true `GEN`) in variable 0.

`GEN ZX_rem(GEN x, GEN y)` returns the remainder of the Euclidean division of x mod y. Assume that x, y are two `ZX` and that y is monic.

`GEN ZX_mod_Xnm1(GEN T, ulong n)` return T modulo $X^n - 1$). Shallow function.

`GEN ZX_div_by_X_1(GEN T, GEN *r)` return the quotient of T by $X - 1$. If r is not `NULL` set it to $T(1)$.

`GEN ZX_digits(GEN x, GEN B)` returns a vector of `ZX` $[c_0, \ldots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^{n} c_i B^i$. Assume that B is monic.

`GEN ZXV_ZX_fromdigits(GEN v, GEN B)` where $v = [c_0, \ldots, c_n]$ is a vector of `ZX`, returns $\sum_{i=0}^{n} c_i B^i$.

`GEN ZX_gcd(GEN x, GEN y)` returns a gcd of the `ZX` x and y. Not memory-clean, but suitable for `gerepileupto`.

`GEN ZX_gcd_all(GEN x, GEN y, GEN *pX)` returns a gcd d of x and y. If `pX` is not `NULL`, set `*pX` to a (nonzero) integer multiple of x/d. If x and y are both monic, then d is monic and `*pX` is exactly x/d. Not memory clean.

`GEN ZX_radical(GEN x)` returns the largest squarefree divisor of the `ZX` x. Not memory clean.

`GEN ZX_content(GEN x)` returns the content of the `ZX` x.

`long ZX_val(GEN P)` as `RgX_val`, but assumes `P` has `t_INT` coefficients.

`long ZX_valrem(GEN P, GEN *z)` as `RgX_valrem`, but assumes `P` has `t_INT` coefficients.

`GEN ZX_to_monic(GEN q GEN *L)` given q a nonzero `ZX`, returns a monic integral polynomial Q such that $Q(x) = Cq(x/L)$, for some rational C and positive integer $L > 0$. If `L` is not `NULL`, set `*L` to L; if $L = 1$, `*L` is set to `gen_1`. Shallow function.

`GEN ZX_primitive_to_monic(GEN q, GEN *L)` as `ZX_to_monic` except q is assumed to have trivial content, which avoids recomputing it. The result is suboptimal if q is not primitive (L larger than necessary), but remains correct. Shallow function.

`GEN ZX_Z_normalize(GEN q, GEN *L)` a restricted version of `ZX_primitive_to_monic`, where q is a *monic* `ZX` of degree > 0. Finds the largest integer $L > 0$ such that $Q(X) := L^{-\deg q}q(Lx)$ is integral and return Q; this is not well-defined if q is a monomial, in that case, set $L = 1$ and $Q = q$. If `L` is not `NULL`, set `*L` to L. Shallow function.

`GEN ZX_Q_normalize(GEN q, GEN *L)` a variant of `ZX_Z_normalize` where $L > 0$ is allowed to be rational, the monic $Q \in \mathbf{Z}[X]$ has possibly smaller coefficients. Shallow function.

`GEN ZX_Q_mul(GEN x, GEN y)` returns $x * y$, where y is a rational number and the resulting `t_POL` has rational entries.

`long ZX_deflate_order(GEN P)` given a nonconstant `ZX` P, returns the largest exponent d such that P is of the form $P(x^d)$.

`long ZX_deflate_max(GEN P, long *d)`. Given a nonconstant polynomial with integer coefficients P, sets `d` to `ZX_deflate_order(P)` and returns `RgX_deflate(P,d)`. Shallow function.

`GEN ZX_rescale(GEN P, GEN h)` returns $h^{\deg(P)}P(x/h)$. `P` is a `ZX` and `h` is a nonzero integer. Neither memory-clean nor suitable for `gerepileupto`.

`GEN ZX_rescale2n(GEN P, long n)` returns $2^{n\deg(P)}P(x >> n)$ where `P` is a `ZX`.

`GEN ZX_rescale_lt(GEN P)` returns the monic integral polynomial $h^{\deg(P)-1}P(x/h)$, where `P` is a nonzero `ZX` and `h` is its leading coefficient. Neither memory-clean nor suitable for `gerepileupto`.

`GEN ZX_translate(GEN P, GEN c)` assume P is a `ZX` and c an integer. Returns $P(X+c)$ (optimized for $c = \pm 1$).

`GEN ZX_affine(GEN P, GEN a, GEN b)` P is a `ZX`, a and b are `t_INT`. Return $P(aX+b)$ (optimized for $b = \pm 1$). Not memory clean.

`GEN ZX_Z_eval(GEN P, GEN x)` evaluate the `ZX` P at the integer x.

`GEN ZX_unscale(GEN P, GEN h)` given a `ZX` P and a `t_INT` h, returns $P(hx)$. Not memory clean.

`GEN ZX_z_unscale(GEN P, long h)` given a `ZX` P, returns $P(hx)$. Not memory clean.

`GEN ZX_unscale2n(GEN P, long n)` given a `ZX` P, returns $P(x << n)$. Not memory clean.

GEN ZX_unscale_div(GEN P, GEN h) given a ZX P and a t_INT h such that $h \mid P(0)$, returns $P(hx)/h$. Not memory clean.

GEN ZX_unscale_divpow(GEN P, GEN h, long k) given a ZX P, a t_INT h and $k > 0$, returns $P(hx)/h^k$ assuming the result has integral coefficients. Not memory clean.

GEN ZX_eval1(GEN P) returns the integer $P(1)$.

GEN ZX_graeffe(GEN p) returns the Graeffe transform of p, i.e. the ZX q such that $p(x)p(-x) = q(x^2)$.

GEN ZX_deriv(GEN x) returns the derivative of x.

GEN ZX_resultant(GEN A, GEN B) returns the resultant of the ZX A and B.

GEN ZX_disc(GEN T) returns the discriminant of the ZX T.

GEN ZX_factor(GEN T) returns the factorization of the primitive part of T over $\mathbf{Q}[X]$ (the content is lost).

int ZX_is_squarefree(GEN T) returns 1 if the ZX T is squarefree, 0 otherwise.

long ZX_is_irred(GEN T) returns 1 it T is irreducible, and 0 otherwise.

GEN ZX_squff(GEN T, GEN *E) write $T(x)$ as a product $\prod T_i^{e_i}$ with the $e_1 < e_2 < \cdots$ all distinct and the T_i pairwise coprime. Return the vector of the T_i, and set *E to the vector of the e_i, as a t_VECSMALL. For efficiency, powers of x should have been removed from T using ZX_valrem, but the result is also correct if not. Not memory clean.

GEN ZX_Uspensky(GEN P, GEN ab, long flag, long bitprec) let P be a ZX polynomial whose real roots are simple and bitprec is the relative precision in bits. For efficiency reasons, P should not only have simple real roots but actually be primitive and squarefree, but the routine neither checks nor enforces this, and it returns correct results in this case as well.

- If flag is 0 returns a list of intervals that isolate the real roots of P. The return value is a column of elements which are either vectors [a,b] of rational numbers meaning that there is a single nonrational root in the open interval (a,b) or elements x0 such that x0 is a rational root of P. Beware that the limits of the open intervals can be roots of the polynomial.
- If flag is 1 returns an approximation of the real roots of P.
- If flag is 2 returns the number of roots.

The argument ab specify the interval in which the roots are searched. The default interval is $(-\infty, \infty)$. If ab is an integer or fraction a then the interval is $[a, \infty)$. If ab is a vector $[a, b]$, where t_INT, t_FRAC or t_INFINITY are allowed for a and b, the interval is $[a, b]$.

long ZX_sturm(GEN P) number of real roots of the nonconstant squarefree ZX P. For efficiency, it is advised to make P primitive first.

long ZX_sturmpart(GEN P, GEN ab) number of real roots of the nonconstant squarefree ZX P in the interval specified by ab: either NULL (no restriction) or a t_VEC $[a, b]$ with two real components (of type t_INT, t_FRAC or t_INFINITY). For efficiency, it is advised to make P primitive first.

long ZX_sturm_irred(GEN P) number of real roots of the ZX P, assumed irreducible over $\mathbf{Q}[X]$. For efficiency, it is advised to make P primitive first.

long ZX_realroots_irred(GEN P, long prec) real roots of the ZX P, assumed irreducible over $\mathbf{Q}[X]$ to precision prec. For efficiency, it is advised to make P primitive first.

7.6.2 Resultants.

GEN ZX_ZXY_resultant(GEN A, GEN B) under the assumption that A in $\mathbf{Z}[Y]$, B in $\mathbf{Q}[Y][X]$, and $R = \text{Res}_Y(A, B) \in \mathbf{Z}[X]$, returns the resultant R.

GEN ZX_compositum_disjoint(GEN A, GEN B) given two irreducible ZX defining linearly disjoint extensions, returns a ZX defining their compositum.

GEN ZX_compositum(GEN A, GEN B, long *lambda) given two irreducible ZX, returns an irreducible ZX C defining their compositum and set lambda to a small integer k such that if α is a root of A and β is a root of B, then $k\alpha + \beta$ is a root of C.

GEN ZX_ZXY_rnfequation(GEN A, GEN B, long *lambda), assume A in $\mathbf{Z}[Y]$, B in $\mathbf{Q}[Y][X]$, and $R = \text{Res}_Y(A, B) \in \mathbf{Z}[X]$. If lambda = NULL, returns R as in ZY_ZXY_resultant. Otherwise, lambda must point to some integer, e.g. 0 which is used as a seed. The function then finds a small $\lambda \in \mathbf{Z}$ (starting from *lambda) such that $R_\lambda(X) := \text{Res}_Y(A, B(X + \lambda Y))$ is squarefree, resets *lambda to the chosen value and returns R_λ.

7.6.3 ZXV.

GEN ZXV_equal(GEN x, GEN y) returns 1 if the two vectors of ZX are equal, as per ZX_equal (variables are not checked to be equal) and 0 otherwise.

GEN ZXV_Z_mul(GEN x, GEN y) multiplies the vector of ZX x by the integer y.

GEN ZXV_remi2n(GEN x, long n) applies ZX_remi2n to all coefficients of x.

GEN ZXV_dotproduct(GEN x, GEN y) as RgV_dotproduct assuming x and y have ZX entries.

7.6.4 ZXT.

GEN ZXT_remi2n(GEN x, long n) applies ZX_remi2n to all leaves of the tree x.

7.6.5 ZXQ.

GEN ZXQ_mul(GEN x, GEN y, GEN T) returns $x * y \bmod T$, assuming that all inputs are ZXs and that T is monic.

GEN ZXQ_sqr(GEN x, GEN T) returns $x^2 \bmod T$, assuming that all inputs are ZXs and that T is monic.

GEN ZXQ_powu(GEN x, ulong n, GEN T) returns $x^n \bmod T$, assuming that all inputs are ZXs and that T is monic.

GEN ZXQ_powers(GEN x, long n, GEN T) returns $[\mathrm{x}^0, \ldots, \mathrm{x}^\mathrm{n}] \bmod T$ as a t_VEC, assuming that all inputs are ZXs and that T is monic.

GEN ZXQ_charpoly(GEN A, GEN T, long v): let T and A be ZXs, returns the characteristic polynomial of Mod(A, T). More generally, A is allowed to be a QX, hence possibly has rational coefficients, *assuming* the result is a ZX, i.e. the algebraic number Mod(A,T) is integral over Z.

GEN ZXQ_minpoly(GEN A, GEN B, long d, ulong bound) let T and A be ZXs, returns the minimal polynomial of Mod(A, T) assuming it has degree d and its coefficients are less than 2^{bound}. More generally, A is allowed to be a QX, hence possibly has rational coefficients, *assuming* the result is a ZX, i.e. the algebraic number Mod(A,T) is integral over Z.

7.6.6 ZXn.

GEN ZXn_mul(GEN x, GEN y, long n) return $xy \pmod{X^n}$.

GEN ZXn_sqr(GEN x, long n) return $x^2 \pmod{X^n}$.

GEN eta_ZXn(long r, long n) return $\eta(X^r) = \prod_{i>0}(1 - X^{ri}) \pmod{X^n}$, $r > 0$.

GEN eta_product_ZXn(GEN DR, long n): DR $= [D, R]$ being a vector with two t_VECSMALL components, return $\prod_i \eta(X^{d_i})^{r_i}$. Shallow function.

7.6.7 ZXQM.

ZXQM are matrices of ZXQ. All entries must be integers or polynomials of degree strictly less than the degree of T.

GEN ZXQM_mul(GEN x, GEN y, GEN T) returns $x * y$ mod T, assuming that all inputs are ZXs and that T is monic.

GEN ZXQM_sqr(GEN x, GEN T) returns x^2 mod T, assuming that all inputs are ZXs and that T is monic.

7.6.8 ZXQX.

GEN ZXQX_mul(GEN x, GEN y, GEN T) returns $x * y$, assuming that all inputs are ZXQXs and that T is monic.

GEN ZXQX_ZXQ_mul(GEN x, GEN y, GEN T) returns $x * y$, assuming that x is a ZXQX, y is a ZXQ and T is monic.

GEN ZXQX_sqr(GEN x, GEN T) returns x^2, assuming that all inputs are ZXQXs and that T is monic.

GEN ZXQX_gcd(GEN x, GEN y, GEN T) returns the gcd of x and y, assuming that all inputs are ZXQXs and that T is monic.

7.6.9 ZXX.

void RgX_check_ZXX(GEN x, const char *s) Assuming x is a t_POL raise an error if it one of its coefficients is not an integer or a ZX (s should point to the name of the caller).

GEN ZXX_renormalize(GEN x, long l), as normalizepol, where l = lg(x), in place.

long ZXX_max_lg(GEN x) returns the effective length of the longest component in x; assume all coefficients are t_INT or ZXs.

GEN ZXX_evalx0(GEN P) returns $P(X, 0)$.

GEN ZXX_Z_mul(GEN x, GEN y) returns xy.

GEN ZXX_Q_mul(GEN x, GEN y) returns $x * y$, where y is a rational number and the resulting t_POL has rational entries.

GEN ZXX_Z_add_shallow(GEN x, GEN y) returns $x + y$. Shallow function.

GEN ZXX_Z_divexact(GEN x, GEN y) returns x/y assuming all integer divisions are exact.

GEN Kronecker_to_ZXX(GEN z, long n, long v) recover $P(X, Y)$ from its Kronecker form $P(X, X^{2n-1})$ (see RgXX_to_Kronecker), v is the variable number corresponding to Y. Shallow function.

`GEN Kronecker_to_ZXQX(GEN z, GEN T)`. Let $n = \deg T$ and let $P(X,Y) \in \mathbf{Z}[X,Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{Z}[X]/(T)$ and $\deg_X P < 2n-1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P (see `RgXX_to_Kronecker`), this function returns $Q \in \mathbf{Z}[X,t]$ such that Q is congruent to $P(X,t)$ mod $(T(X))$, $\deg_X Q < n$. Not stack-clean. Note that t need not be the same variable as Y!

`GEN ZXX_mul_Kronecker(GEN P, GEN Q, long n)` return `ZX_mul` applied to the Kronecker forms $P(X, X^{2n-1})$ and $Q(X, X^{2n-1})$ of P and Q. Not memory clean.

`GEN ZXX_sqr_Kronecker(GEN P, long n)` return `ZX_sqr` applied to the Kronecker forms $P(X, X^{2n-1})$ of P. Not memory clean.

7.6.10 `QX`.

`void RgX_check_QX(GEN x, const char *s)` Assuming `x` is a `t_POL` raise an error if it is not a `QX` (s should point to the name of the caller).

`GEN QX_mul(GEN x, GEN y)`

`GEN QX_sqr(GEN x)`

`GEN QX_ZX_rem(GEN x, GEN y)` y is assumed to be monic.

`GEN QX_gcd(GEN x, GEN y)` returns a gcd of the `QX` x and y.

`GEN QX_disc(GEN T)` returns the discriminant of the `QX` `T`.

`GEN QX_factor(GEN T)` as `ZX_factor`.

`GEN QX_resultant(GEN A, GEN B)` returns the resultant of the `QX` `A` and `B`.

`GEN QX_complex_roots(GEN p, long l)` returns the complex roots of the `QX` p at accuracy l, where real roots are returned as `t_REAL`s. More efficient when p is irreducible and primitive. Special case of `cleanroots`.

7.6.11 `QXQ`.

`GEN QXQ_norm(GEN A, GEN B)` A being a `QX` and B being a `ZX`, returns the norm of the algebraic number $A \bmod B$, using a modular algorithm. To ensure that B is a `ZX`, one may replace it by `Q_primpart(B)`, which of course does not change the norm.

If A is not a `ZX` — it has a denominator —, but the result is nevertheless known to be an integer, it is much more efficient to call `QXQ_intnorm` instead.

`GEN QXQ_intnorm(GEN A, GEN B)` A being a `QX` and B being a `ZX`, returns the norm of the algebraic number $A \bmod B$, *assuming* that the result is an integer, which is for instance the case is $A \bmod B$ is an algebraic integer, in particular if A is a `ZX`. To ensure that B is a `ZX`, one may replace it by `Q_primpart(B)` (which of course does not change the norm).

If the result is not known to be an integer, you must use `QXQ_norm` instead, which is slower.

`GEN QXQ_mul(GEN A, GEN B, GEN T)` returns the product of A and B modulo T where both A and B are a `QX` and T is a monic `ZX`.

`GEN QXQ_sqr(GEN A, GEN T)` returns the square of A modulo T where A is a `QX` and T is a monic `ZX`.

`GEN QXQ_inv(GEN A, GEN B)` returns the inverse of A modulo B where A is a `QX` and B is a `ZX`. Should you need this for a `QX` B, just use

```
QXQ_inv(A, Q_primpart(B));
```

But in all cases where modular arithmetic modulo B is desired, it is much more efficient to replace B by `Q_primpart`(B) once and for all.

`GEN QXQ_div(GEN A, GEN B, GEN T)` returns A/B modulo T where A and B are `QX` and T is a `ZX`. Use this function when the result is expected to be of the same size as $B^{-1} \bmod T$ or smaller. Otherwise, it will be faster to use `QXQ_mul(A,QXQ_inv(B,T),T)`.

`GEN QXQ_charpoly(GEN A, GEN T, long v)` where `A` is a `QX` and `T` is a `ZX`, returns the characteristic polynomial of `Mod(A, T)`. If the result is known to be a `ZX`, then calling `ZXQ_charpoly` will be faster.

`GEN QXQ_powers(GEN x, long n, GEN T)` returns $[\mathrm{x}^0, \ldots, \mathrm{x}^\mathrm{n}]$ as `RgXQ_powers` would, but in a more efficient way when x has a huge integer denominator (we start by removing that denominator). Assume that x is a `QX` and T is a `ZX`. Meant to precompute powers of algebraic integers in $\mathbf{Q}[t]/(T)$.

`GEN QXQ_reverse(GEN f, GEN T)` as `RgXQ_reverse`, assuming f is a `QX`.

`GEN QX_ZXQV_eval(GEN f, GEN nV, GEN dV)` as `RgX_RgXQV_eval`, except that f is assumed to be a `QX`, V is given implicitly by a numerator `nV` (`ZV`) and denominator `dV` (a positive `t_INT` or `NULL` for trivial denominator). Not memory clean, but suitable for `gerepileupto`.

`GEN QXV_QXQ_eval(GEN v, GEN a, GEN T)` v is a vector of `QX`s (possibly scalars, i.e. rational numbers, for convenience), a and T both `QX`. Return the vector of evaluations at a modulo T. Not memory clean, nor suitable for `gerepileupto`.

`GEN QXY_QXQ_evalx(GEN P, GEN a, GEN T)` $P(X,Y)$ is a `t_POL` with `QX` coefficients (possibly scalars, i.e. rational numbers, for convenience) , a and T both `QX`. Return the `QX` $P(a \bmod T, Y)$. Not memory clean, nor suitable for `gerepileupto`.

7.6.12 QXQX.

`GEN QXQX_mul(GEN x, GEN y, GEN T)` where T is a monic `ZX`.

`GEN QXQX_QXQ_mul(GEN x, GEN y, GEN T)` where T is a monic `ZX`.

`GEN QXQX_sqr(GEN x, GEN T)` where T is a monic `ZX`

`GEN QXQX_powers(GEN x, long n, GEN T)` where T is a monic `ZX`

`GEN nfgcd(GEN P, GEN Q, GEN T, GEN den)` given P and Q in $\mathbf{Z}[X,Y]$, T monic irreducible in $\mathbf{Z}[Y]$, returns the primitive d in $\mathbf{Z}[X,Y]$ which is a gcd of P, Q in $K[X]$, where K is the number field $\mathbf{Q}[Y]/(T)$. If not `NULL`, `den` is a multiple of the integral denominator of the (monic) gcd of P, Q in $K[X]$.

`GEN nfgcd_all(GEN P, GEN Q, GEN T, GEN den, GEN *Pnew)` as `nfgcd`. If `Pnew` is not `NULL`, set `*Pnew` to a nonzero integer multiple of P/d. If P and Q are both monic, then d is monic and `*Pnew` is exactly P/d. Not memory clean if the gcd is 1 (in that case `*Pnew` is set to P).

`GEN QXQX_gcd(GEN x, GEN y, GEN T)` returns the gcd of x and y, assuming that x and y are `QXQX`s and that T is a monic `ZX`.

`GEN QXQX_homogenous_evalpow(GEN P, GEN a, GEN B, GEN T)` Evaluate the homogenous polynomial associated to the univariate polynomial P on (a,b) where B is the vector of powers of b with exponents 0 to the degree of P (`QXQ_powers(b,degpol(P),T)`).

7.6.13 QXQM.

QXQM are matrices of QXQ. All entries must be t_INT, t_FRAC or polynomials of degree strictly less than the degree of T, which must be a monic ZX.

GEN QXQM_mul(GEN x, GEN y, GEN T) returns $x * y \bmod T$.

GEN QXQM_sqr(GEN x, GEN T) returns $x^2 \bmod T$.

7.6.14 zx.

GEN zero_zx(long sv) returns a zero zx in variable v.

GEN polx_zx(long sv) returns the variable v as degree 1 Flx.

GEN zx_renormalize(GEN x, long l), as Flx_renormalize, where l = lg(x), in place.

GEN zx_shift(GEN T, long n) return T multiplied by x^n, assuming $n \geq 0$.

long zx_lval(GEN f, long p) return the valuation of f at p.

GEN zx_z_divexact(GEN x, long y) return x/y assuming all divisions are exact.

7.6.15 RgX.

7.6.15.1 Tests.

long RgX_degree(GEN x, long v) x being a t_POL and $v \geq 0$, returns the degree in v of x. Error if x is not a polynomial in v.

int RgX_isscalar(GEN x) return 1 if x all the coefficients of x of degree > 0 are 0 (as per gequal0).

int RgX_is_rational(GEN P) return 1 if the RgX P has only rational coefficients (t_INT and t_FRAC), and 0 otherwise.

int RgX_is_QX(GEN P) return 1 if the RgX P has only t_INT and t_FRAC coefficients, and 0 otherwise.

int RgX_is_ZX(GEN P) return 1 if the RgX P has only t_INT coefficients, and 0 otherwise.

int RgX_is_monomial(GEN x) returns 1 (true) if x is a nonzero monomial in its main variable, 0 otherwise.

long RgX_equal(GEN x, GEN y) returns 1 if the t_POLs x and y have the same degpol and their coefficients are equal (as per gequal). Variable numbers are not checked. Note that this is more stringent than gequal(x,y), which only checks whether $x - y$ satisfies gequal0; in particular, they may have different apparent degrees provided the extra leading terms are 0.

long RgX_equal_var(GEN x, GEN y) returns 1 if x and y have the same variable number and RgX_equal(x,y) is 1.

7.6.15.2 Coefficients, blocks.

`GEN RgX_coeff(GEN P, long n)` return the coefficient of x^n in P, defined as `gen_0` if $n < 0$ or $n >$ `degpol`(P). Shallow function.

`int RgX_blocks(GEN P, long n, long m)` writes $P(X) = a_0(X) + X^n * a_1(X) * X^n + \ldots + X^{n*(m-1)} a_{m-1}(X)$, where the a_i are polynomial of degree at most $n-1$ (except possibly for the last one) and returns $[a_0(X), a_1(X), \ldots, a_{m-1}(X)]$. Shallow function.

`void RgX_even_odd(GEN p, GEN *pe, GEN *po)` write $p(X) = E(X^2) + XO(X^2)$ and set `*pe = E`, `*po = O`. Shallow function.

`GEN RgX_splitting(GEN P, long k)` write $P(X) = a_0(X^k) + Xa_1(X^k) + \ldots + X^{k-1}a_{k-1}(X^k)$ and return $[a_0(X), a_1(X), \ldots, a_{k-1}(X)]$. Shallow function.

`GEN RgX_copy(GEN x)` returns (a deep copy of) `x`.

`GEN RgX_renormalize(GEN x)` remove leading terms in `x` which are equal to (necessarily inexact) zeros.

`GEN RgX_renormalize_lg(GEN x, long lx)` as `setlg(x, lx)` followed by `RgX_renormalize(x)`. Assumes that `lx` $\leq$ `lg`(x).

`GEN RgX_recip(GEN P)` returns the reverse of the polynomial P, i.e. $X^{\deg P} P(1/X)$.

`GEN RgX_recip_shallow(GEN P)` shallow function of `RgX_recip`.

`GEN RgX_recip_i(GEN P)` shallow function of `RgX_recip`, where we further assume that $P(0) \neq 0$, so that the degree of the output is the degree of P.

`long rfracrecip(GEN *a, GEN *b)` let `*a` and `*b` be such that their quotient F is a `t_RFRAC` in variable X. Write $F(1/X) = X^v A/B$ where A and B are coprime to X and v in $\mathbf{Z}$. Set `*a` to A, `*b` to B and return v.

`GEN RgX_deflate(GEN P, long d)` assuming P is a polynomial of the form $Q(X^d)$, return Q. Shallow function, not suitable for `gerepileupto`.

`long RgX_deflate_order(GEN P)` given a nonconstant polynomial P, returns the largest exponent d such that P is of the form $P(x^d)$ (use `gequal0` to check whether coefficients are 0).

`long RgX_deflate_max(GEN P, long *d)` given a nonconstant polynomial P, sets `d` to `RgX_deflate_order(P)` and returns `RgX_deflate(P,d)`. Shallow function.

`long rfrac_deflate_order(GEN F)` as `RgX_deflate_order` where F is a nonconstant `t_RFRAC`.

`long rfrac_deflate_max(GEN F, long *d)` as `RgX_deflate_max` where F is a nonconstant `t_RFRAC`.

`GEN rfrac_deflate(GEN F, long m)` as `RgX_deflate` where F is a `t_RFRAC`.

`GEN RgX_inflate(GEN P, long d)` return $P(X^d)$. Shallow function, not suitable for `gerepileupto`.

`GEN RgX_rescale_to_int(GEN x)` given a polynomial x with real entries (`t_INT`, `t_FRAC` or `t_REAL`), return a `ZX` wich is very close to Dx for some well-chosen integer D. More precisely, if the input is exact, D is the denominator of x; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

`GEN RgX_homogenize(GEN P, long v)` Return the homogenous polynomial associated to P in the secondary variable v, that is $y^d * P(x/y)$ where d is the degree of P, x is the variable of P, and y is the variable with number v.

`GEN RgX_homogenous_evalpow(GEN P, GEN a, GEN B)` Evaluate the homogenous polynomial associated to the univariate polynomial P on (a, b) where B is the vector of powers of b with exponents 0 to the degree of P (`gpowers(b,degpol(P))`).

`GEN RgXX_to_Kronecker(GEN P, long n)` Assuming $P(X, Y)$ is a polynomial of degree in X strictly less than n, returns $P(X, X^{2*n-1})$, the Kronecker form of P. Shallow function.

`GEN RgXX_to_Kronecker_spec(GEN Q, long lQ, long n)` return `RgXX_to_Kronecker`(P, n), where P is the polynomial $\sum_{i=0}^{\mathtt{lQ}-1} Q[i]x^i$. To be used when splitting the coefficients of genuine polynomials into blocks. Shallow function.

7.6.15.3 Shifts, valuations.

`GEN RgX_shift(GEN x, long n)` returns $\mathtt{x} * t^n$ if $n \geq 0$, and $\mathtt{x} \backslash t^{-n}$ otherwise.

`GEN RgX_shift_shallow(GEN x, long n)` as `RgX_shift`, but shallow (coefficients are not copied).

`GEN RgX_rotate_shallow(GEN P, long k, long p)` returns $\mathtt{P} * X^k \pmod{X^p - 1}$, assuming the degree of P is strictly less than p, and $k \geq 0$.

`void RgX_shift_inplace_init(long v)` $v \geq 0$, prepare for a later call to `RgX_shift_inplace`. Reserves v words on the stack.

`GEN RgX_shift_inplace(GEN x, long v)` $v \geq 0$, assume that `RgX_shift_inplace_init`(v) has been called (reserving v words on the stack), immediately followed by a `t_POL` x. Return `RgX_shift`(x, v) by shifting x in place. To be used as follows

```
RgX_shift_inplace_init(v);
av = avma;
...
x = gerepileupto(av, ...); /* a t_POL */
return RgX_shift_inplace(x, v);
```

`long RgX_valrem(GEN P, GEN *pz)` returns the valuation v of the `t_POL P` with respect to its main variable X. Check whether coefficients are 0 using `isexactzero`. Set `*pz` to `RgX_shift_shallow`$(P, -v)$.

`long RgX_val(GEN P)` returns the valuation v of the `t_POL P` with respect to its main variable X. Check whether coefficients are 0 using `isexactzero`.

`long RgX_valrem_inexact(GEN P, GEN *z)` as `RgX_valrem`, using `gequal0` instead of `isexactzero`.

`long RgXV_maxdegree(GEN V)` returns the maximum of the degrees of the components of the vector of `t_POL`s V.

7.6.15.4 Basic arithmetic.

`GEN RgX_add(GEN x, GEN y)` adds `x` and `y`.

`GEN RgX_sub(GEN x, GEN y)` subtracts `x` and `y`.

`GEN RgX_neg(GEN x)` returns −`x`.

`GEN RgX_Rg_add(GEN y, GEN x)` returns $x + y$.

`GEN RgX_Rg_add_shallow(GEN y, GEN x)` returns $x + y$; shallow function.

`GEN Rg_RgX_sub(GEN x, GEN y)`

`GEN RgX_Rg_sub(GEN y, GEN x)` returns $x - y$

`GEN RgX_Rg_mul(GEN y, GEN x)` multiplies the `RgX` `y` by the scalar `x`.

`GEN RgX_muls(GEN y, long s)` multiplies the `RgX` `y` by the `long` `s`.

`GEN RgX_mul2n(GEN y, long n)` multiplies the `RgX` `y` by 2^n.

`GEN RgX_Rg_div(GEN y, GEN x)` divides the `RgX` `y` by the scalar `x`.

`GEN RgX_divs(GEN y, long s)` divides the `RgX` `y` by the `long` `s`.

`GEN RgX_Rg_divexact(GEN x, GEN y)` exact division of the `RgX` `y` by the scalar `x`.

`GEN RgX_Rg_eval_bk(GEN f, GEN x)` returns `f`(`x`) using Brent and Kung algorithm. (Use `poleval` for Horner algorithm.)

`GEN RgX_RgV_eval(GEN f, GEN V)` as `RgX_Rg_eval_bk(f, x)`, assuming V was output by `gpowers(x, n)` for some $n \geq 1$.

`GEN RgXV_RgV_eval(GEN f, GEN V)` apply `RgX_RgV_eval_bk(, V)` to all the components of the vector f.

`GEN RgX_normalize(GEN x)` divides x by its leading coefficient. If the latter is 1, x itself is returned, not a copy. Leading coefficients equal to 0 are stripped, e.g.

```
0.*t^3 + Mod(0,3)*t^2 + 2*t
```

is normalized to t.

`GEN RgX_mul(GEN x, GEN y)` multiplies the two `t_POL` (in the same variable) `x` and `y`. Detect the coefficient ring and use an appropriate algorithm.

`GEN RgX_mul_i(GEN x, GEN y)` multiplies the two `t_POL` (in the same variable) `x` and `y`. Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

`GEN RgX_mul_normalized(GEN A, long a, GEN B, long b)` returns $(X^a+A)(X^b+B)-X^(a+b)$, where we assume that $\deg A < a$ and $\deg B < b$ are polynomials in the same variable X.

`GEN RgX_sqr(GEN x)` squares the `t_POL` `x`. Detect the coefficient ring and use an appropriate algorithm.

`GEN RgX_sqr_i(GEN x)` squares the `t_POL` `x`. Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

`GEN RgXV_prod(GEN V)`, `V` being a vector of `RgX`, returns their product.

GEN RgX_divrem(GEN x, GEN y, GEN *r) by default, returns the Euclidean quotient and store the remainder in r. Three special values of r change that behavior • NULL: do not store the remainder, used to implement RgX_div,

- ONLY_REM: return the remainder, used to implement RgX_rem,
- ONLY_DIVIDES: return the quotient if the division is exact, and NULL otherwise.

In the generic case, the remainder is created after the quotient and can be disposed of individually with a cgiv(r).

GEN RgX_div(GEN x, GEN y)

GEN RgX_div_by_X_x(GEN A, GEN a, GEN *r) returns the quotient of the RgX A by ($X -$ a), and sets r to the remainder A(a).

GEN RgX_rem(GEN x, GEN y)

GEN RgX_pseudodivrem(GEN x, GEN y, GEN *ptr) compute a pseudo-quotient q and pseudo-remainder r such that $\texttt{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$. Return q and set *ptr to r.

GEN RgX_pseudorem(GEN x, GEN y) return the remainder in the pseudo-division of x by y.

GEN RgXQX_pseudorem(GEN x, GEN y, GEN T) return the remainder in the pseudo-division of x by y over $R[X]/(T)$.

int ZXQX_dvd(GEN x, GEN y, GEN T) let T be a monic irreducible ZX, let x, y be t_POL whose coefficients are either t_INTs or ZX in the same variable as T. Assume further that the leading coefficient of y is an integer. Return 1 if $y|x$ in $(\mathbf{Z}[Y]/(T))[X]$, and 0 otherwise.

GEN RgXQX_pseudodivrem(GEN x, GEN y, GEN T, GEN *ptr) compute a pseudo-quotient q and pseudo-remainder r such that $\texttt{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$ in $R[X]/(T)$. Return q and set *ptr to r.

GEN RgX_mulXn(GEN a, long n) returns $a * X^n$. This may be a t_FRAC if $n < 0$ and the valuation of a is not large enough.

GEN RgX_addmulXn(GEN a, GEN b, long n) returns $a + b * X^n$, assuming that $n > 0$.

GEN RgX_addmulXn_shallow(GEN a, GEN b, long n) shallow variant of RgX_addmulXn.

GEN RgX_digits(GEN x, GEN B) returns a vector of RgX $[c_0, \ldots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^{n} c_i B^i$.

7.6.15.5 Internal routines working on coefficient arrays.

These routines operate on coefficient blocks which are invalid GENs A GEN argument a or b in routines below is actually a coefficient arrays representing the polynomials $\sum_{i=0}^{\texttt{na}-1} a[i]X^i$ and $\sum_{i=0}^{\texttt{nb}-1} b[i]X^i$. Note that $a[0]$ and $b[0]$ contain coefficients and not the mandatory GEN codeword. This allows to implement divide-and-conquer methods directly, without needing to allocate wrappers around coefficient blocks.

GEN RgX_mulspec(GEN a, GEN b, long na, long nb). Internal routine: given two coefficient arrays representing polynomials, return their product (as a true GEN) in variable 0.

GEN RgX_sqrspec(GEN a, long na). Internal routine: given a coefficient array representing a polynomial r eturn its square (as a true GEN) in variable 0.

GEN RgX_addspec(GEN x, GEN y, long nx, long ny) given two coefficient arrays representing polynomials, return their sum (as a true GEN) in variable 0.

GEN RgX_addspec_shallow(GEN x, GEN y, long nx, long ny) shallow variant of RgX_addspec.

7.6.15.6 GCD, Resultant.

`GEN RgX_gcd(GEN x, GEN y)` returns the GCD of `x` and `y`, assumed to be `t_POL`s in the same variable.

`GEN RgX_gcd_simple(GEN x, GEN y)` as `RgX_gcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_gcd`.

`GEN RgX_extgcd(GEN x, GEN y, GEN *u, GEN *v)` returns $d = \text{GCD}(\mathtt{x},\mathtt{y})$, and sets `*u`, `*v` to the Bezout coefficients such that $*\mathtt{ux} + *\mathtt{vy} = d$. Uses a generic subresultant algorithm.

`GEN RgX_extgcd_simple(GEN x, GEN y, GEN *u, GEN *v)` as `RgX_extgcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_extgcd`.

`GEN RgX_halfgcd(GEN x, GEN y)` assuming `x` and `y` are `t_POL`s in the same variable, returns a 2-components `t_VEC` $[M, V]$ where M is a 2×2 `t_MAT` and V a 2-component `t_COL`, both with `t_POL` entries, such that $M*[x, y] \;== V$ and such that f $V = [a, b]$, then $\deg a \geq \lceil \max(\deg x, \deg y))/2\rceil > \deg b$.

`GEN RgX_chinese_coprime(GEN x, GEN y, GEN Tx, GEN Ty, GEN Tz)` returns an `RgX`, congruent to `x` mod `Tx` and to `y` mod `Ty`. Assumes `Tx` and `Ty` are coprime, and `Tz = Tx * Ty` or `NULL` (in which case it is computed within).

`GEN RgX_disc(GEN x)` returns the discriminant of the `t_POL` `x` with respect to its main variable.

`GEN RgX_resultant_all(GEN x, GEN y, GEN *sol)` returns `resultant(x,y)`. If `sol` is not `NULL`, sets it to the last nonconstant remainder in the polynomial remainder sequence if it exists and to `gen_0` otherwise (e.g. one polynomial has degree 0).

7.6.15.7 Other operations.

`GEN RgX_gtofp(GEN x, GEN prec)` returns the polynomial obtained by applying

```
gtofp(gel(x,i), prec)
```

to all coefficients of x.

`GEN RgX_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

```
gnorml2( RgX_gtofp(x, prec) )
```

`GEN RgX_deriv(GEN x)` returns the derivative of `x` with respect to its main variable.

`GEN RgX_integ(GEN x)` returns the primitive of `x` vanishing at 0, with respect to its main variable.

`GEN RgX_rescale(GEN P, GEN h)` returns $h^{\deg(P)}P(x/h)$. `P` is an `RgX` and `h` is nonzero. (Leaves small objects on the stack. Suitable but inefficient for `gerepileupto`.)

`GEN RgX_unscale(GEN P, GEN h)` returns $P(hx)$. (Leaves small objects on the stack. Suitable but inefficient for `gerepileupto`.)

`GEN RgXV_unscale(GEN v, GEN h)` apply `RgX_unscale` to a vector of `RgX`.

`GEN RgX_translate(GEN P, GEN c)` assume c is a scalar or a polynomials whose main variable has lower priority than the main variable X of P. Returns $P(X + c)$ (optimized for $c = \pm 1$).

`GEN RgX_affine(GEN P, GEN a, GEN b)` Return $P(aX + b)$ (optimized for $b = \pm 1$). Not memory clean.

7.6.15.8 Function related to modular forms.

`GEN RgX_act_Gl2Q(GEN g, long k)` let R be a commutative ring and $g = [a, b; c, d]$ be in $\mathrm{GL}_2(\mathbf{Q})$, g acts (on the left) on homogeneous polynomials of degree $k-2$ in $V := R[X, Y]_{k-2}$ via

$$g \cdot P := P(dX - cY, -bX + aY) = (\det g)^{k-2} P((X, Y) \cdot g^{-1}).$$

This function returns the matrix in $M_{k-1}(R)$ of $P \mapsto g \cdot P$ in the basis $(X^{k-2}, \dots, Y^{k-2})$ of V.

`GEN RgX_act_ZGl2Q(GEN z, long k)` let $G := \mathrm{GL}_2(\mathbf{Q})$, acting on $R[X, Y]_{k-2}$ and $z \in \mathbf{Z}[G]$. Return the matrix giving $P \mapsto z \cdot P$ in the basis $(X^{k-2}, \dots, Y^{k-2})$.

7.6.16 RgXn.

`GEN RgXn_red_shallow(GEN x, long n)` return `x`$\%t^n$, where $n \geq 0$. Shallow function.

`GEN RgXn_recip_shallow(GEN P)` returns $X^n P(1/X)$. Shallow function.

`GEN RgXn_mul(GEN a, GEN b, long n)` returns ab modulo X^n, where a, b are two `t_POL` in the same variable X and $n \geq 0$. Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

`GEN RgXn_sqr(GEN a, long n)` returns a^2 modulo X^n, where a is a `t_POL` in the variable X and $n \geq 0$. Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

`GEN RgX_mulhigh_i(GEN f, GEN g, long n)` return the Euclidean quotient of $f(x) * g(x)$ by x^n (high product). Uses `RgXn_mul` applied to the reciprocal polynomials of f and g. Not suitable for `gerepile`.

`GEN RgX_sqrhigh_i(GEN f, long n)` return the Euclidean quotient of $f(x)^2$ by x^n (high product). Uses `RgXn_sqr` applied to the reciprocal polynomial of f. Not suitable for `gerepile`.

`GEN RgXn_inv(GEN a, long n)` returns a^{-1} modulo X^n, where a is a `t_POL` in the variable X and $n \geq 0$. Uses Newton-Raphson algorithm.

`GEN RgXn_inv_i(GEN a, long n)` as `RgXn_inv` without final garbage collection (suitable for `gerepileupto`).

`GEN RgXn_div(GEN a, GEN b, long n)` returns a/b modulo X^n, where a and b are `t_POL`s in the variable X and $n \geq 0$. Uses Newton-Raphson/Karp-Markstein algorithm.

`GEN RgXn_div_i(GEN a, GEN b, long n)` as `RgXn_div` without final garbage collection (suitable for `gerepileupto`).

`GEN RgXn_powers(GEN x, long m, long n)` returns $[\mathtt{x}^0, \dots, \mathtt{x}^{\mathtt{m}}]$ modulo X^n as a `t_VEC` of `RgXn`s.

`GEN RgXn_powu(GEN x, ulong m, long n)` returns x^m modulo X^n.

`GEN RgXn_powu_i(GEN x, ulong m, long n)` as `RgXn_powu`, not memory clean.

`GEN RgXn_sqrt(GEN a, long n)` returns $a^{1/2}$ modulo X^n, where a is a `t_POL` in the variable X and $n \geq 0$. Assume that $a = 1 \bmod X$. Uses Newton algorithm.

`GEN RgXn_exp(GEN a, long n)` returns $exp(a)$ modulo X^n, assuming $a = 0 \bmod X$.

`GEN RgXn_expint(GEN f, long n)` return $\exp(F)$ where F is the primitive of f that vanishes at 0.

`GEN RgXn_eval(GEN Q, GEN x, long n)` special case of `RgX_RgXQ_eval`, when the modulus is a monomial: returns `Q(x)` modulo t^n, where $x \in R[t]$.

`GEN RgX_RgXn_eval(GEN f, GEN x, long n)` returns `f(x)` modulo X^n.

`GEN RgX_RgXnV_eval(GEN f, GEN V, long n)` as `RgX_RgXn_eval(f, x, n)`, assuming V was output by `RgXn_powers(x, m, n)` for some $m \geq 1$.

`GEN RgXn_reverse(GEN f, long n)` assuming that $f = ax \bmod x^2$ with a invertible, returns a `t_POL` g of degree $< n$ such that $(g \circ f)(x) = x$ modulo x^n.

7.6.17 `RgXnV`.

`GEN RgXnV_red_shallow(GEN x, long n)` apply `RgXn_red_shallow` to all the components of the vector x.

7.6.18 `RgXQ`.

`GEN RgXQ_mul(GEN y, GEN x, GEN T)` computes xy mod T

`GEN RgXQ_sqr(GEN x, GEN T)` computes x^2 mod T

`GEN RgXQ_inv(GEN x, GEN T)` return the inverse of x mod T.

`GEN RgXQ_pow(GEN x, GEN n, GEN T)` computes x^n mod T

`GEN RgXQ_powu(GEN x, ulong n, GEN T)` computes x^n mod T, n being an `ulong`.

`GEN RgXQ_powers(GEN x, long n, GEN T)` returns $[\mathrm{x}^0, \ldots, \mathrm{x}^\mathrm{n}]$ as a `t_VEC` of `RgXQ`s.

`GEN RgXQ_matrix_pow(GEN y, long n, long m, GEN P)` returns `RgXQ_powers(y,m-1,P)`, as a matrix of dimension $n \geq \deg P$.

`GEN RgXQ_norm(GEN x, GEN T)` returns the norm of `Mod(x, T)`.

`GEN RgXQ_trace(GEN x, GEN T)` returns the trace of `Mod(x, T)`.

`GEN RgXQ_charpoly(GEN x, GEN T, long v)` returns the characteristic polynomial of `Mod(x, T)`, in variable v.

`GEN RgXQ_minpoly(GEN x, GEN T, long v)` returns the minimal polynomial of `Mod(x, T)`, in variable v.

`GEN RgX_RgXQ_eval(GEN f, GEN x, GEN T)` returns `f(x)` modulo T.

`GEN RgX_RgXQV_eval(GEN f, GEN V, GEN T)` as `RgX_RgXQ_eval(f, x, T)`, assuming V was output by `RgXQ_powers(x, n, T)` for some $n \geq 1$.

`int RgXQ_ratlift(GEN x, GEN T, long amax, long bmax, GEN *P, GEN *Q)` Assuming that $\mathtt{amax} + \mathtt{bmax} < \deg T$, attempts to recognize x as a rational function a/b, i.e. to find `t_POL`s P and Q such that

- $P \equiv Qx$ modulo T,
- $\deg P \leq \mathtt{amax}$, $\deg Q \leq \mathtt{bmax}$,
- $\gcd(T, P) = \gcd(P, Q)$.

If unsuccessful, the routine returns 0 and leaves P, Q unchanged; otherwise it returns 1 and sets P and Q.

`GEN RgXQ_reverse(GEN f, GEN T)` returns a `t_POL` g of degree $< n = \deg T$ such that $T(x)$ divides $(g \circ f)(x) - x$, by solving a linear system. Low-level function underlying `modreverse`: it returns a lift of [modreverse(f,T)]; faster than the high-level function since it needs not compute the characteristic polynomial of f mod T (often already known in applications). In the trivial case where $n \leq 1$, returns a scalar, not a constant `t_POL`.

7.6.19 `RgXQV`, `RgXQC`.

`GEN RgXQC_red(GEN z, GEN T)` `z` a vector whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise) in a `t_COL`.

`GEN RgXQV_red(GEN z, GEN T)` `z` a vector whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise) in a `t_VEC`.

`GEN RgXQV_RgXQ_mul(GEN z, GEN x, GEN T)` `z` multiplies the `RgXQV` `z` by the scalar (`RgXQ`) `x`.

`GEN RgXQV_factorback(GEN L, GEN e, GEN T)` returns $\prod_i L_i^{e_i} \bmod T$ where L is a vector of `RgXQ`s and e a vector of `t_INT`s.

7.6.20 `RgXQM`.

`GEN RgXQM_red(GEN z, GEN T)` `z` a matrix whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise).

`GEN RgXQM_mul(GEN x, GEN y, GEN T)`

7.6.21 `RgXQX`.

`GEN RgXQX_red(GEN z, GEN T)` `z` a `t_POL` whose coefficients are `RgX`s (arbitrary `GEN`s in fact), reduce them to `RgXQ`s (applying `grem` coefficientwise).

`GEN RgXQX_mul(GEN x, GEN y, GEN T)`

`GEN RgXQX_RgXQ_mul(GEN x, GEN y, GEN T)` multiplies the `RgXQX` `y` by the scalar (`RgXQ`) `x`.

`GEN RgXQX_sqr(GEN x, GEN T)`

`GEN RgXQX_powers(GEN x, long n, GEN T)`

`GEN RgXQX_divrem(GEN x, GEN y, GEN T, GEN *pr)`

`GEN RgXQX_div(GEN x, GEN y, GEN T)`

`GEN RgXQX_rem(GEN x, GEN y, GEN T)`

`GEN RgXQX_translate(GEN P, GEN c, GEN T)` assume the main variable X of P has higher priority than the main variable Y of T and c. Return a lift of $P(X + \mathrm{Mod}(c(Y), T(Y)))$.

`GEN Kronecker_to_mod(GEN z, GEN T)` $z \in R[X]$ represents an element $P(X,Y)$ in $R[X,Y]$ mod $T(Y)$ in Kronecker form, i.e. $z = P(X, X^{2*n-1})$

Let R be some commutative ring, $n = \deg T$ and let $P(X,Y) \in R[X,Y]$ lift a polynomial in $K[Y]$, where $K := R[X]/(T)$ and $\deg_X P < 2n-1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P, this function returns the image of $P(X,t)$ in $K[t]$, with `t_POLMOD` coefficients. Not stack-clean. Note that t need not be the same variable as Y!

Chapter 8:
Black box algebraic structures

The generic routines like `gmul` or `gadd` allow handling objects belonging to a fixed list of basic types, with some natural polymorphism (you can mix rational numbers and polynomials, etc.), at the expense of efficiency and sometimes of clarity when the recursive structure becomes complicated, e.g. a few levels of t_POLMODs attached to different polynomials and variable numbers for quotient structures. This is the only possibility in GP.

On the other hand, the Level 2 Kernel allows dedicated routines to handle efficiently objects of a very specific type, e.g. polynomials with coefficients in the same finite field. This is more efficient, but imvolves a lot of code duplication since polymorphism is no longer possible.

A third and final option, still restricted to library programming, is to define an arbitrary algebraic structure (currently groups, fields, rings, algebras and $\mathbf{Z}_p$-modules) by providing suitable methods, then using generic algorithms. For instance naive Gaussian pivoting applies over all base fields and need only be implemented once. The difference with the first solution is that we no longer depend on the way functions like gmul or gadd will guess what the user is trying to do. We can then implement independently various groups / fields / algebras in a clean way.

8.1 Black box groups.

A black box group is defined by a bb_group struct, describing methods available to handle group elements:

```
struct bb_group
{
  GEN (*mul)(void*, GEN, GEN);
  GEN (*pow)(void*, GEN, GEN);
  GEN (*rand)(void*);
  ulong (*hash)(GEN);
  int (*equal)(GEN, GEN);
  int (*equal1)(GEN);
  GEN (*easylog)(void *E, GEN, GEN, GEN);
};
```

mul(E,x,y) returns the product xy.

pow(E,x,n) returns x^n (n integer, possibly negative or zero).

rand(E) returns a random element in the group.

hash(x) returns a hash value for x (hash_GEN is suitable for this field).

equal(x,y) returns one if $x = y$ and zero otherwise.

equal1(x) returns one if x is the neutral element in the group, and zero otherwise.

easylog(E,a,g,o) (optional) returns either NULL or the discrete logarithm n such that $g^n = a$, the element g being of order o. This provides a short-cut in situation where a better algorithm than the generic one is known.

A group is thus described by a struct bb_group as above and auxiliary data typecast to void*. The following functions operate on black box groups:

GEN gen_Shanks_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)
Generic baby-step/giant-step algorithm (Shanks's method). Assuming that g has order N, compute an integer k such that $g^k = x$. Return cgetg(1, t_VEC) if there are no solutions. This requires $O(\sqrt{N})$ group operations and uses an auxiliary table containing $O(\sqrt{N})$ group elements.

The above is useful for a one-shot computation. If many discrete logs are desired: GEN gen_Shanks_init(GEN g, long n, void *E, const struct bb_group *grp) return an auxiliary data structure T required to compute a discrete log in base g. Compute and store all powers g^i, $i < n$.

GEN gen_Shanks(GEN T, GEN x, ulong N, void *E, const struct bb_group *grp) Let T be computed by gen_Shanks_init$(g, n, \ldots)$. Return $k < nN$ such that $g^k = x$ or NULL if no such index exist. It uses $O(N)$ operation in the group and fast table lookups (in time $O(\log n)$). The interface is such that the function may be used when the order of the base g is unknown, and hence compute it given only an upper bound B for it: e.g. choose n, N such that $nN \geq B$ and compute the discrete log l of g^{-1} in base g, then use gen_order with multiple $N = l + 1$.

GEN gen_Pollard_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp) Generic Pollard rho algorithm. Assuming that g has order N, compute an integer k such that $g^k = x$. This requires $O(\sqrt{N})$ group operations in average and $O(1)$ storage. Will enter an infinite loop if there are no solutions.

GEN gen_plog(GEN x, GEN g, GEN N, void *E, const struct bb_group) Assuming that g has prime order N, compute an integer k such that $g^k = x$, using either gen_Shanks_log or gen_Pollard_log. Return cgetg(1, t_VEC) if there are no solutions.

GEN gen_Shanks_sqrtn(GEN a, GEN n, GEN N, GEN *zetan, void *E, const struct bb_group *grp) returns one solution of $x^n = a$ in a black box cyclic group of order N. Return NULL if no solution exists. If zetan is not NULL it is set to an element of exact order n. This function uses gen_plog for all prime divisors of $\gcd(n, N)$.

GEN gen_PH_log(GEN a, GEN g, GEN N, void *E, const struct bb_group *grp) returns an integer k such that $g^k = x$, assuming that the order of g divides N, using Pohlig-Hellman algorithm. Return cgetg(1, t_VEC) if there are no solutions. This calls gen_plog repeatedly for all prime divisors p of N.

In the following functions the integer parameter ord can be given in all the formats recognized for the argument of arithmetic functions, i.e. either as a positive t_INT N, or as its factorization matrix *faN*, or (preferred) as a pair $[N, faN]$.

GEN gen_order(GEN x, GEN ord, void *E, const struct bb_group *grp) computes the order of x; ord is a multiple of the order, for instance the group order.

GEN gen_factored_order(GEN x, GEN ord, void *E, const struct bb_group *grp) returns a pair $[o, F]$, where o is the order of x and F is the factorization of o; ord is as in gen_order.

GEN gen_gener(GEN ord, void *E, const struct bb_group *grp) returns a random generator of the group, assuming it is of order exactly ord.

GEN get_arith_Z(GEN ord) given ord as above in one of the formats recognized for arithmetic functions, i.e. a positive t_INT N, its factorization *faN*, or the pair $[N, faN]$, return N.

GEN get_arith_ZZM(GEN ord) given ord as above, return the pair $[N, faN]$. This may require factoring N.

GEN gen_select_order(GEN v, void *E, const struct bb_group *grp) Let v be a vector of possible orders for the group; try to find the true order by checking orders of random points. This will not terminate if there is an ambiguity.

8.1.1 Black box groups with pairing.

These functions handle groups of rank at most 2 equipped with a family of bilinear pairings which behave like the Weil pairing on elliptic curves over finite field. In the descriptions below, the function `pairorder(E, P, Q, m, F)` must return the order of the m-pairing of P and Q, both of order dividing m, where F is the factorization matrix of a multiple of m.

`GEN gen_ellgroup(GEN o, GEN d, GEN *pt_m, void *E, const struct bb_group *grp, GEN pairorder(void *E, GEN P, GEN Q, GEN m, GEN F))` returns the elementary divisors $[d_1, d_2]$ of the group, assuming it is of order exactly $o > 1$, and that d_2 divides d. If $d_2 = 1$ then $[o]$ is returned, otherwise `m=*pt_m` is set to the order of the pairing required to verify a generating set which is to be used with `gen_ellgens`. For the parameter o, all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair $[n, \texttt{factor}(n)]$.

`GEN gen_ellgens(GEN d1, GEN d2, GEN m, void *E, const struct bb_group *grp, GEN pairorder(void *E, GEN P, GEN Q, GEN m, GEN F))` the parameters d_1, d_2, m being as returned by `gen_ellgroup`, returns a pair of generators $[P, Q]$ such that P is of order d_1 and the m-pairing of P and Q is of order m. (Note: Q needs not be of order d_2). For the parameter d_1, all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair $[n, \texttt{factor}(n)]$.

8.1.2 Functions returning black box groups.

`const struct bb_group * get_Flxq_star(void **E, GEN T, ulong p)`

`const struct bb_group * get_FpXQ_star(void **E, GEN T, GEN p)` returns a pointer to the black box group $(\mathbf{F}_p[x]/(T))^*$.

`const struct bb_group * get_FpE_group(void **pE, GEN a4, GEN a6, GEN p)` returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group $E(\mathbf{F}_p)$ where E is the elliptic curve $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in $\mathbf{F}_p$.

`const struct bb_group * get_FpXQE_group(void **pE, GEN a4, GEN a6, GEN T, GEN p)` returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group $E(\mathbf{F}_p[X]/(T))$ where E is the elliptic curve $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in $\mathbf{F}_p[X]/(T)$.

`const struct bb_group * get_FlxqE_group(void **pE, GEN a4, GEN a6, GEN T, ulong p)` idem for small p.

`const struct bb_group * get_F2xqE_group(void **pE, GEN a2, GEN a6, GEN T)` idem for $p = 2$.

8.2 Black box fields.

A black box field is defined by a bb_field struct, describing methods available to handle field elements:

```
struct bb_field
{
  GEN (*red)(void *E ,GEN);
  GEN (*add)(void *E ,GEN, GEN);
  GEN (*mul)(void *E ,GEN, GEN);
  GEN (*neg)(void *E ,GEN);
  GEN (*inv)(void *E ,GEN);
  int (*equal0)(GEN);
  GEN (*s)(void *E, long);
};
```

In contrast of black box group, elements can have non canonical forms, and only red is required to return a canonical form. For instance a black box implementation of finite fields, all methods except red may return arbitrary representatives in $\mathbf{Z}[X]$ of the correct congruence class modulo $(p, T(X))$.

red(E,x) returns the canonical form of x.

add(E,x,y) returns the sum $x + y$.

mul(E,x,y) returns the product xy.

neg(E,x) returns $-x$.

inv(E,x) returns the inverse of x.

equal0(x) x being in canonical form, returns one if $x = 0$ and zero otherwise.

s(n) n being a small signed integer, returns n times the unit element.

A field is thus described by a struct bb_field as above and auxiliary data typecast to void*. The following functions operate on black box fields:

GEN gen_Gauss(GEN a, GEN b, void *E, const struct bb_field *ff)

GEN gen_Gauss_pivot(GEN x, long *rr, void *E, const struct bb_field *ff)

GEN gen_det(GEN a, void *E, const struct bb_field *ff)

GEN gen_ker(GEN x, long deplin, void *E, const struct bb_field *ff)

GEN gen_matcolinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)

GEN gen_matcolmul(GEN a, GEN b, void *E, const struct bb_field *ff)

GEN gen_matid(long n, void *E, const struct bb_field *ff)

GEN gen_matinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)

GEN gen_matmul(GEN a, GEN b, void *E, const struct bb_field *ff)

8.2.1 Functions returning black box fields.

`const struct bb_field * get_Fp_field(void **pE, GEN p)`

`const struct bb_field * get_Fq_field(void **pE, GEN T, GEN p)`

`const struct bb_field * get_Flxq_field(void **pE, GEN T, ulong p)`

`const struct bb_field * get_F2xq_field(void **pE, GEN T)`

`const struct bb_field * get_nf_field(void **pE, GEN nf)`

8.3 Black box algebra.

A black box algebra is defined by a `bb_algebra` struct, describing methods available to handle algebra elements:

```
struct bb_algebra
{
  GEN (*red)(void *E, GEN x);
  GEN (*add)(void *E, GEN x, GEN y);
  GEN (*sub)(void *E, GEN x, GEN y);
  GEN (*mul)(void *E, GEN x, GEN y);
  GEN (*sqr)(void *E, GEN x);
  GEN (*one)(void *E);
  GEN (*zero)(void *E);
};
```

In contrast with black box groups, elements can have non canonical forms, but only `add` is allowed to return a non canonical form.

`red(E,x)` returns the canonical form of x.

`add(E,x,y)` returns the sum $x + y$.

`sub(E,x,y)` returns the difference $x - y$.

`mul(E,x,y)` returns the product xy.

`sqr(E,x)` returns the square x^2.

`one(E)` returns the unit element.

`zero(E)` returns the zero element.

An algebra is thus described by a `struct bb_algebra` as above and auxiliary data typecast to `void*`. The following functions operate on black box algebra:

`GEN gen_bkeval(GEN P, long d, GEN x, int use_sqr, void *E, const struct bb_algebra *ff, GEN cmul(void *E, GEN P, long a, GEN x))` x being an element of the black box algebra, and P some black box polynomial of degree d over the base field, returns $P(x)$. The function `cmul(E,P,a,y)` must return the coefficient of degree a of P multiplied by y. `cmul` is allowed to return a non canonical form; it is also allowed to return `NULL` instead of an exact 0.

The flag `use_sqr` has the same meaning as for `gen_powers`. This implements an algorithm of Brent and Kung (1978).

GEN gen_bkeval_powers(GEN P, long d, GEN V, void *E, const struct bb_algebra *ff, GEN cmul(void *E, GEN P, long a, GEN x)) as gen_RgX_bkeval assuming V was output by gen_powers(x, l, E, ff) for some $l \geq 1$. For optimal performance, l should be computed by brent_kung_optpow.

long brent_kung_optpow(long d, long n, long m) returns the optimal parameter l for the evaluation of n/m polynomials of degree d. Fractional values can be used if the evaluations are done with different accuracies, and thus have different weights.

8.3.1 Functions returning black box algebras.

const struct bb_algebra * get_FpX_algebra(void **E, GEN p, long v) return the algebra of polynomials over $\mathbf{F}_p$ in variable v.

const struct bb_algebra * get_FpXQ_algebra(void **E, GEN T, GEN p) return the algebra $\mathbf{F}_p[X]/(T(X))$.

const struct bb_algebra * get_FpXQX_algebra(void **E, GEN T, GEN p, long v) return the algebra of polynomials over $\mathbf{F}_p[X]/(T(X))$ in variable v.

const struct bb_algebra * get_FlxqXQ_algebra(void **E, GEN S, GEN T, ulong p) return the algebra $\mathbf{F}_p[X,Y]/(S(X,Y),T(X))$ (for ulong p).

const struct bb_algebra * get_FpXQXQ_algebra(void **E, GEN S, GEN T, GEN p) return the algebra $\mathbf{F}_p[X,Y]/(S(X,Y),T(X))$.

const struct bb_algebra * get_Rg_algebra(void) return the generic algebra.

8.4 Black box ring.

A black box ring is defined by a bb_ring struct, describing methods available to handle ring elements:

```
struct bb_ring
{
  GEN (*add)(void *E, GEN x, GEN y);
  GEN (*mul)(void *E, GEN x, GEN y);
  GEN (*sqr)(void *E, GEN x);
};
```

add(E,x,y) returns the sum $x + y$.

mul(E,x,y) returns the product xy.

sqr(E,x) returns the square x^2.

GEN gen_fromdigits(GEN v, GEN B, void *E, struct bb_ring *r) where B is a ring element and $v = [c_0, \ldots, c_{n-1}]$ a vector of ring elements, return $\sum_{i=0}^{n} c_i B^i$ using binary splitting.

GEN gen_digits(GEN x, GEN B, long n, void *E, struct bb_ring *r, GEN (*div)(void *E, GEN x, GEN y, GEN *r))

(Require the ring to be Euclidean)

div(E,x,y,&r) performs the Euclidean division of x by y in the ring R, returning the quotient q and setting r to the residue so that $x = qy + r$ holds. The residue must belong to a fixed set of representatives of $R/(y)$.

The argument x being a ring element, gen_digits returns a vector of ring elements $[c_0, \ldots, c_{n-1}]$ such that $x = \sum_{i=0}^{n} c_i B^i$. Furthermore for all $i \neq n-1$, the elements c_i belonging to the fixed set of representatives of $R/(B)$.

8.5 Black box free $\mathbf{Z}_p$-modules.

(Very experimental)

`GEN gen_ZpX_Dixon(GEN F, GEN V, GEN q, GEN p, long N, void *E, GEN lin(void *E, GEN F, GEN z, GEN q), GEN invl(void *E, GEN z))`

Let F be a ZpXT representing the coefficients of some abstract linear mapping f over $\mathbf{Z}_p[X]$ seen as a free $\mathbf{Z}_p$-module, let V be an element of $\mathbf{Z}_p[X]$ and let $q = p^N$. Return $y \in \mathbf{Z}_p[X]$ such that $f(y) = V \pmod{p^N}$ assuming the following holds for $n \leq N$:

- $\texttt{lin}(E, \texttt{FpX\_red}(F, p^n), z, p^n) \equiv f(z) \pmod{p^n}$
- $f(\texttt{invl}(E, z)) \equiv z \pmod{p}$

The rationale for the argument F being that it allows gen_ZpX_Dixon to reduce it to the required p-adic precision.

`GEN gen_ZpX_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q), GEN invd(void *E, GEN b, GEN v, GEN q, long N))`

Let x be an element of $\mathbf{Z}_p[X]$ seen as a free $\mathbf{Z}_p$-module, and f some differentiable function over $\mathbf{Z}_p[X]$ such that $f(x) \equiv 0 \pmod{p}$. Return y such that $f(y) \equiv 0 \pmod{p^n}$, assuming the following holds for all $a, b \in \mathbf{Z}_p[X]$ and $M \leq N$:

- $v = \texttt{eval}(E, a, p^N)$ is a vector of elements of $\mathbf{Z}_p[X]$,
- $w = \texttt{invd}(E, b, v, p^M, M)$ is an element in $\mathbf{Z}_p[X]$,
- $v[1] \equiv f(a) \pmod{p^N \mathbf{Z}_p[X]}$,
- $df_a(w) \equiv b \pmod{p^M \mathbf{Z}_p[X]}$

and df_a denotes the differential of f at a. Motivation: eval allows to evaluate f and invd allows to invert its differential. Frequently, data useful to compute the differential appear as a subproduct of computing the function. The vector v allows eval to provide these to invd. The implementation of invd will generally involves the use of the function gen_ZpX_Dixon.

`GEN gen_ZpM_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q), GEN invd(void *E, GEN b, GEN v, GEN q, long N))` as above, with polynomials replaced by matrices.

Chapter 9:
Operations on general PARI objects

9.1 Assignment.

It is in general easier to use a direct conversion, e.g. `y = stoi(s)`, than to allocate a target of correct type and sufficient size, then assign to it:

```
GEN y = cgeti(3); affsi(s, y);
```

These functions can still be moderately useful in complicated garbage collecting scenarios but you will be better off not using them.

`void gaffsg(long s, GEN x)` assigns the `long s` into the object `x`.

`void gaffect(GEN x, GEN y)` assigns the object `x` into the object `y`. Both `x` and `y` must be scalar types. Type conversions (e.g. from `t_INT` to `t_REAL` or `t_INTMOD`) occur if legitimate.

`int is_universal_constant(GEN x)` returns 1 if x is a global PARI constant you should never assign to (such as `gen_1`), and 0 otherwise.

9.2 Conversions.

9.2.1 Scalars.

`double rtodbl(GEN x)` applied to a `t_REAL x`, converts `x` into a `double` if possible.

`GEN dbltor(double x)` converts the `double x` into a `t_REAL`.

`long dblexpo(double x)` returns `expo(dbltor(x))`, but faster and without cluttering the stack.

`ulong dblmantissa(double x)` returns the most significant word in the mantissa of `dbltor(x)`.

`int gisdouble(GEN x)` if `x` is a real number (not necessarily a `t_REAL`), return 1 if `x` can be converted to a `double`, 0 otherwise.

`double gtodouble(GEN x)` if `x` is a real number (not necessarily a `t_REAL`), converts `x` into a `double` if possible.

`long gtos(GEN x)` converts the `t_INT x` to a small integer if possible, otherwise raise an exception. This function is similar to `itos`, slightly slower since it checks the type of `x`.

`ulong gtou(GEN x)` converts the non-negative `t_INT x` to an unsigned small integer if possible, otherwise raise an exception. This function is similar to `itou`, slightly slower since it checks the type of `x`.

`double dbllog2r(GEN x)` assuming that `x` is a nonzero `t_REAL`, returns an approximation to `log2(|x|)`.

`double dblmodulus(GEN x)` return an approximation to `|x|`.

`long gtolong(GEN x)` if `x` is an integer (not necessarily a `t_INT`), converts `x` into a `long` if possible.

`GEN fractor(GEN x, long l)` applied to a `t_FRAC` `x`, converts `x` into a `t_REAL` of length `prec`.

`GEN quadtofp(GEN x, long l)` applied to a `t_QUAD` `x`, converts `x` into a `t_REAL` or `t_COMPLEX` depending on the sign of the discriminant of `x`, to precision `l` `BITS_IN_LONG`-bit words.

`GEN upper_to_cx(GEN x, long *prec)` valid for a `t_COMPLEX` or `t_QUAD` belonging to the upper half-plane. If a `t_QUAD`, convert it to `t_COMPLEX` using accuracy `*prec`. If x is inexact, sets `*prec` to the precision of x.

`GEN cxtofp(GEN x, long prec)` converts the `t_COMPLEX` `x` to a a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`.

`GEN cxcompotor(GEN x, long prec)` converts the `t_INT`, `t_REAL` or `t_FRAC` x to a `t_REAL` of length `prec`. These are all the real types which may occur as components of a `t_COMPLEX`; special case of `gtofp` (introduced so that the latter is not recursive and can thus be inlined).

`GEN cxtoreal(GEN x)` converts the complex (`t_INT`, `t_REAL`, `t_FRAC` or `t_COMPLEX`) x to a real number if its imaginary part is 0. Shallow function.

converts the `t_COMPLEX` `x` to a a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`.

`GEN gtofp(GEN x, long prec)` converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of precision `prec`; not necessarily of *length* `prec`: a real 0 may be given as `real_0(...)`). If the result is a `t_COMPLEX` extra care is taken so that its modulus really has accuracy `prec`: there is a problem if the real part of the input is an exact 0; indeed, converting it to `real_0(prec)` would be wrong if the imaginary part is tiny, since the modulus would then become equal to 0, as in $1.E-100+0.E-28 = 0.E-28$.

`GEN gtomp(GEN z, long prec)` converts the real number `x` (`t_INT`, `t_REAL`, `t_FRAC`, real `t_QUAD`) to either a `t_INT` or a `t_REAL` of precision `prec`. Not memory clean if x is a `t_INT`: we return x itself and not a copy.

`GEN gcvtop(GEN x, GEN p, long l)` converts x into a `t_PADIC` of precision l. Works componentwise on recursive objects, e.g. `t_POL` or `t_VEC`. Converting 0 yields $O(p^l)$; converting a nonzero number yield a result well defined modulo $p^{v_p(x)+l}$.

`GEN cvtop(GEN x, GEN p, long l)` as `gcvtop`, assuming that x is a scalar.

`GEN cvtop2(GEN x, GEN y)` y being a p-adic, converts the scalar x to a p-adic of the same accuracy. Shallow function.

`GEN cvstop2(long s, GEN y)` y being a p-adic, converts the scalar s to a p-adic of the same accuracy. Shallow function.

`GEN gprec(GEN x, long l)` returns a copy of x whose precision is changed to l digits. The precision change is done recursively on all components of x. Digits means *decimal*, p-adic and X-adic digits for `t_REAL`, `t_SER`, `t_PADIC` components, respectively.

`GEN gprec_w(GEN x, long l)` returns a shallow copy of x whose `t_REAL` components have their precision changed to l *words*. This is often more useful than `gprec`.

`GEN gprec_wtrunc(GEN x, long l)` returns a shallow copy of x whose `t_REAL` components have their precision *truncated* to l *words*. Contrary to `gprec_w`, this function may never increase the precision of x.

GEN gprec_wensure(GEN x, long l) returns a shallow copy of x whose t_REAL components have their precision *increased* to at least l *words*. Contrary to gprec_w, this function may never decrease the precision of x.

The following functions are obsolete and kept for backward compatibility only:

GEN precision0(GEN x, long n)

GEN bitprecision0(GEN x, long n)

9.2.2 Modular objects / lifts.

GEN gmodulo(GEN x, GEN y) creates the object **Mod**(x,y) on the PARI stack, where x and y are either both t_INTs, and the result is a t_INTMOD, or x is a scalar or a t_POL and y a t_POL, and the result is a t_POLMOD.

GEN gmodulgs(GEN x, long y) same as **gmodulo** except y is a long.

GEN gmodulsg(long x, GEN y) same as **gmodulo** except x is a long.

GEN gmodulss(long x, long y) same as **gmodulo** except both x and y are longs.

GEN lift_shallow(GEN x) shallow version of lift

GEN liftall_shallow(GEN x) shallow version of liftall

GEN liftint_shallow(GEN x) shallow version of liftint

GEN liftpol_shallow(GEN x) shallow version of liftpol

GEN centerlift0(GEN x, long v) DEPRECATED, kept for backward compatibility only: use either lift0(x, v) or centerlift(x).

9.2.3 Between polynomials and coefficient arrays.

GEN gtopoly(GEN x, long v) converts or truncates the object x into a t_POL with main variable number v. A common application would be the conversion of coefficient vectors (coefficients are given by decreasing degree). E.g. [2,3] goes to 2*v + 3

GEN gtopolyrev(GEN x, long v) converts or truncates the object x into a t_POL with main variable number v, but vectors are converted in reverse order compared to gtopoly (coefficients are given by increasing degree). E.g. [2,3] goes to 3*v + 2. In other words the vector represents a polynomial in the basis $(1, v, v^2, v^3, \ldots)$.

GEN normalizepol(GEN x) applied to an unnormalized t_POL x (with all coefficients correctly set except that leading_term(x) might be zero), normalizes x correctly in place and returns x. For internal use. Normalizing means deleting all leading *exact* zeroes (as per isexactzero), except if the polynomial turns out to be 0, in which case we try to find a coefficient c which is a nonrational zero, and return the constant polynomial c. (We do this so that information about the base ring is not lost.)

GEN normalizepol_lg(GEN x, long l) applies normalizepol to x, pretending that lg(x) is l, which must be less than or equal to lg(x). If equal, the function is equivalent to normalizepol(x).

GEN normalizepol_approx(GEN x, long lx) as normalizepol_lg, with the difference that we just delete all leading zeroes (as per gequal0). This rougher normalization is used when we have no other choice, for instance before attempting a Euclidean division by x.

The following routines do *not* copy coefficients on the stack (they only move pointers around), hence are very fast but not suitable for gerepile calls. Recall that an RgV (resp. an RgX, resp. an RgM) is a t_VEC or t_COL (resp. a t_POL, resp. a t_MAT) with arbitrary components. Similarly, an RgXV is a t_VEC or t_COL with RgX components, etc.

GEN RgV_to_RgX(GEN x, long v) converts the RgV x to a (normalized) polynomial in variable v (as gtopolyrev, without copy).

GEN RgV_to_RgX_reverse(GEN x, long v) converts the RgV x to a (normalized) polynomial in variable v (as gtopoly, without copy).

GEN RgX_to_RgC(GEN x, long N) converts the t_POL x to a t_COL v with N components. Coefficients of x are listed by increasing degree, so that y[i] is the coefficient of the term of degree $i-1$ in x.

GEN Rg_to_RgC(GEN x, long N) as RgX_to_RgV, except that other types than t_POL are allowed for x, which is then considered as a constant polynomial.

GEN RgM_to_RgXV(GEN x, long v) converts the RgM x to a t_VEC of RgX, by repeated calls to RgV_to_RgX.

GEN RgM_to_RgXV_reverse(GEN x, long v) converts the RgM x to a t_VEC of RgX, by repeated calls to RgV_to_RgX_reverse.

GEN RgV_to_RgM(GEN v, long N) converts the vector v to a t_MAT with N rows, by repeated calls to Rg_to_RgV.

GEN RgXV_to_RgM(GEN v, long N) converts the vector of RgX v to a t_MAT with N rows, by repeated calls to RgX_to_RgV.

GEN RgM_to_RgXX(GEN x, long v, long w) converts the RgM x into a t_POL in variable v, whose coefficients are t_POLs in variable w. This is a shortcut for

```
RgV_to_RgX( RgM_to_RgXV(x, w), v );
```

There are no consistency checks with respect to variable priorities: the above is an invalid object if $\texttt{varncmp}(\texttt{v},\texttt{w}) \geq 0$.

GEN RgXX_to_RgM(GEN x, long N) converts the t_POL x with RgX (or constant) coefficients to a matrix with N rows.

long RgXY_degreex(GEN P) return the degree of P with respect to the secondary variable.

GEN RgXY_derivx(GEN P) return the derivative of P with respect to the secondary variable.

GEN RgXY_swap(GEN P, long n, long w) converts the bivariate polynomial $\texttt{P}(u,v)$ (a t_POL with t_POL or scalar coefficients) to $P(\texttt{pol\_x}[\texttt{w}],u)$, assuming n is an upper bound for $\deg_v(\texttt{P})$.

GEN RgXY_swapspec(GEN C, long n, long w, long lP) as RgXY_swap where the coefficients of P are given by gel(C,0),...,gel(C,lP-1).

GEN RgX_to_ser(GEN x, long l) convert the t_POL x to a *shallow* t_SER of length $l \geq 2$. Unless the polynomial is an exact zero, the coefficient of lowest degree T^d of the result is not an exact zero (as per isexactzero). The remainder is $O(T^{d+l-2})$.

GEN RgX_to_ser_inexact(GEN x, long l) convert the t_POL x to a *shallow* t_SER of length $l \geq 2$. Unless the polynomial is zero, the coefficient of lowest degree T^d of the result is not zero (as per gequal0). The remainder is $O(T^{d+l-2})$.

`GEN RgV_to_ser(GEN x, long v, long l)` convert the `t_VEC` `x`, to a *shallow* `t_SER` of length $l \geq 2$.

`GEN rfrac_to_ser(GEN F, long l)` applied to a `t_RFRAC` F, creates a `t_SER` of length $l \geq 2$ congruent to F. Not memory-clean but suitable for `gerepileupto`.

`GEN rfrac_to_ser_i(GEN F, long l)` internal variant of `rfrac_to_ser`, neither memory-clean nor suitable for `gerepileupto`.

`GEN rfracrecip_to_ser_absolute(GEN F, long d)` applied to a `t_RFRAC` F, creates the `t_SER` $F(1/t) + O(t^d)$. Note that we use absolute and not relative precision here.

`GEN gtoser(GEN s, long v, long d)`. This function is deprecated, kept for backward compatibility: it follows the semantic of `Ser(s,v)`, with $d =$ `seriesprecision` implied and is hard to use as a general conversion function. Use `gtoser_prec` instead.

It converts the object s into a `t_SER` with main variable number `v` and $d > 0$ significant terms, but the argument d is sometimes ignored. More precisely

• if s is a scalar (with respect to variable v), we return a constant power series with d significant terms;

• if s is a `t_POL` in variable v, it is truncated to d terms if needed;

• if s is a vector, the coefficients of the vector are understood to be the coefficients of the power series starting from the constant term (as in `Polrev`), and the precision d is *ignored*;

• if s is already a power series in v, we return a copy, and the precision d is again *ignored*.

`GEN gtoser_prec(GEN s, long v, long d)` this function is a variant of `gtoser` following the semantic of `Ser(s,v,d)`: the precision d is always taken into account.

`GEN gtocol(GEN x)` converts the object `x` into a `t_COL`

`GEN gtomat(GEN x)` converts the object `x` into a `t_MAT`.

`GEN gtovec(GEN x)` converts the object `x` into a `t_VEC`.

`GEN gtovecsmall(GEN x)` converts the object `x` into a `t_VECSMALL`.

`GEN normalizeser(GEN x)` applied to an unnormalized `t_SER` `x` (i.e. type `t_SER` with all coefficients correctly set except that `x[2]` might be zero), normalizes `x` correctly in place. Returns `x`. For internal use.

`GEN serchop0(GEN s)` given a `t_SER` of the form $x^v s(x)$, with $s(0) \neq 0$, return $x^v(s - s(0))$. Shallow function.

`GEN serchop_i(GEN x, long n)` returns a shallow chopy of `t_SER` x with all terms of degree strictly less than n removed. Shallow version of `serchop`.

9.3 Constructors.

9.3.1 Clean constructors.

GEN zeropadic(GEN p, long n) creates a 0 t_PADIC equal to $O(\mathtt{p}^{\mathtt{n}})$.

GEN zeroser(long v, long n) creates a 0 t_SER in variable v equal to $O(X^{\mathtt{n}})$.

GEN scalarser(GEN x, long v, long prec) creates a constant t_SER in variable v and precision prec, whose constant coefficient is (a copy of) x, in other words $\mathtt{x} + O(\mathtt{v}^{\mathtt{prec}})$. Assumes that prec ≥ 0.

GEN pol_0(long v) Returns the constant polynomial 0 in variable v.

GEN pol_1(long v) Returns the constant polynomial 1 in variable v.

GEN pol_x(long v) Returns the monomial of degree 1 in variable v.

GEN pol_xn(long n, long v) Returns the monomial of degree n in variable v; assume that $n \geq 0$.

GEN pol_xnall(long n, long v) Returns the Laurent monomial of degree n in variable v; $n < 0$ is allowed.

GEN pol_x_powers(long N, long v) returns the powers of pol_x(v), of degree 0 to $N-1$, in a vector with N components.

GEN scalarpol(GEN x, long v) creates a constant t_POL in variable v, whose constant coefficient is (a copy of) x.

GEN deg1pol(GEN a, GEN b, long v) creates the degree 1 t_POL $a$pol_x$(v) + b$

GEN zeropol(long v) is identical pol_0.

GEN zerocol(long n) creates a t_COL with n components set to gen_0.

GEN zerovec(long n) creates a t_VEC with n components set to gen_0.

GEN zerovec_block(long n) as zerovec but return a clone.

GEN col_ei(long n, long i) creates a t_COL with n components set to gen_0, but for the i-th one which is set to gen_1 (i-th vector in the canonical basis).

GEN vec_ei(long n, long i) creates a t_VEC with n components set to gen_0, but for the i-th one which is set to gen_1 (i-th vector in the canonical basis).

GEN trivial_fact(void) returns the trivial (empty) factorization Mat([]~,[]~)

GEN prime_fact(GEN x) returns the factorization Mat([x]~, [1]~)

GEN Rg_col_ei(GEN x, long n, long i) creates a t_COL with n components set to gen_0, but for the i-th one which is set to x.

GEN vecsmall_ei(long n, long i) creates a t_VECSMALL with n components set to 0, but for the i-th one which is set to 1 (i-th vector in the canonical basis).

GEN scalarcol(GEN x, long n) creates a t_COL with n components set to gen_0, but the first one which is set to a copy of x. (The name comes from RgV_isscalar.)

GEN mkintmodu(ulong x, ulong y) creates the t_INTMOD Mod(x, y). The inputs must satisfy $x < y$.

GEN zeromat(long m, long n) creates a t_MAT with m x n components set to gen_0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns. To fully allocate a matrix initialized with zero entries, use zeromatcopy.

GEN zeromatcopy(long m, long n) creates a t_MAT with m x n components set to gen_0.

GEN matid(long n) identity matrix in dimension n (with components gen_1 andgen_0).

GEN scalarmat(GEN x, long n) scalar matrix, x times the identity.

GEN scalarmat_s(long x, long n) scalar matrix, stoi(x) times the identity.

GEN vecrange(GEN a, GEN b) returns the t_VEC $[a..b]$.

GEN vecrangess(long a, long b) returns the t_VEC $[a..b]$.

See also next section for analogs of the following functions:

GEN mkfracss(long x, long y) creates the t_FRAC x/y. Assumes that $y > 1$ and $(x, y) = 1$.

GEN sstoQ(long x, long y) returns the t_INT or t_FRAC x/y; no assumptions.

GEN uutoQ(ulong x, ulong y) returns the t_INT or t_FRAC x/y; no assumptions.

void Qtoss(GEN q, long *n, long *d) given a t_INT or t_FRAC q, set n and d such that $q = n/d$ with $d \geq 1$ and $(n, d) = 1$. Overflow error if numerator or denominator do not fit into a long integer.

GEN mkfraccopy(GEN x, GEN y) creates the t_FRAC x/y. Assumes that $y > 1$ and $(x, y) = 1$.

GEN mkrfraccopy(GEN x, GEN y) creates the t_RFRAC x/y. Assumes that y is a t_POL, x a compatible type whose variable has lower or same priority, with $(x, y) = 1$.

GEN mkcolcopy(GEN x) creates a 1-dimensional t_COL containing x.

GEN mkmatcopy(GEN x) creates a 1-by-1 t_MAT wrapping the t_COL x.

GEN mkveccopy(GEN x) creates a 1-dimensional t_VEC containing x.

GEN mkvec2copy(GEN x, GEN y) creates a 2-dimensional t_VEC equal to [x,y].

GEN mkcols(long x) creates a 1-dimensional t_COL containing stoi(x).

GEN mkcol2s(long x, long y) creates a 2-dimensional t_COL containing [stoi(x), stoi(y)] .

GEN mkcol3s(long x, long y, long z) creates a 3-dimensional t_COL containing [stoi(x), stoi(y), stoi(z)] .

GEN mkcol4s(long x, long y, long z, long t) creates a 4-dimensional t_COL containing [stoi(x), stoi(y), stoi(z), stoi(t)] .

GEN mkvecs(long x) creates a 1-dimensional t_VEC containing stoi(x).

GEN mkvec2s(long x, long y) creates a 2-dimensional t_VEC containing [stoi(x), stoi(y)].

GEN mkmat22s(long a, long b, long c, long d) creates the 2 by 2 t_MAT with successive rows [stoi(a), stoi(b)] and [stoi(c), stoi(d)].

GEN mkvec3s(long x, long y, long z) creates a 3-dimensional t_VEC containing [stoi(x), stoi(y), stoi(z)].

GEN mkvec4s(long x, long y, long z, long t) creates a 4-dimensional t_VEC containing [stoi(x), stoi(y), stoi(z), stoi(t)].

GEN mkvecsmall(long x) creates a 1-dimensional t_VECSMALL containing x.

GEN mkvecsmall2(long x, long y) creates a 2-dimensional t_VECSMALL containing [x, y].

GEN mkvecsmall3(long x, long y, long z) creates a 3-dimensional t_VECSMALL containing [x, y, z].

GEN mkvecsmall4(long x, long y, long z, long t) creates a 4-dimensional t_VECSMALL containing [x, y, z, t].

GEN mkvecsmall5(long x, long y, long z, long t, long u) creates a 5-dimensional t_VECSMALL containing [x, y, z, t, u].

GEN mkvecsmalln(long n, ...) returns the t_VECSMALL whose n coefficients (long) follow. *Warning:* since this is a variadic function, C type promotion is not performed on the arguments by the compiler, thus you have to make sure that all the arguments are of type long, in particular integer constants need to be written with the L suffix: mkvecsmalln(2, 1L, 2L) is correct, but mkvecsmalln(2, 1, 2) is not.

9.3.2 Unclean constructors.

Contrary to the policy of general PARI functions, the functions in this subsection do *not* copy their arguments, nor do they produce an object a priori suitable for gerepileupto. In particular, they are faster than their clean equivalent (which may not exist). *If* you restrict their arguments to universal objects (e.g gen_0), then the above warning does not apply.

GEN mkcomplex(GEN x, GEN y) creates the t_COMPLEX $x + iy$.

GEN mulcxI(GEN x) creates the t_COMPLEX ix. The result in general contains data pointing back to the original x. Use gcopy if this is a problem. But in most cases, the result is to be used immediately, before x is subject to garbage collection.

GEN mulcxmI(GEN x), as mulcxI, but returns $-ix$.

GEN mulcxpowIs(GEN x, long k), as mulcxI, but returns $x \cdot i^k$.

GEN mkquad(GEN n, GEN x, GEN y) creates the t_QUAD $x + yw$, where w is a root of n, which is of the form quadpoly(D).

GEN quadpoly_i(GEN D) creates the canonical quadratic polynomial of discriminant D. Assume that the t_INT D is congruent to $0, 1$ mod 4 and not a square.

GEN mkfrac(GEN x, GEN y) creates the t_FRAC x/y. Assumes that $y > 1$ and $(x, y) = 1$.

GEN mkrfrac(GEN x, GEN y) creates the t_RFRAC x/y. Assumes that y is a t_POL, x a compatible type whose variable has lower or same priority, with $(x, y) = 1$.

GEN mkcol(GEN x) creates a 1-dimensional t_COL containing x.

GEN mkcol2(GEN x, GEN y) creates a 2-dimensional t_COL equal to [x,y].

GEN mkcol3(GEN x, GEN y, GEN z) creates a 3-dimensional t_COL equal to [x,y,z].

GEN mkcol4(GEN x, GEN y, GEN z, GEN t) creates a 4-dimensional t_COL equal to [x,y,z,t].

GEN mkcol5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5) creates the 5-dimensional t_COL equal to $[a_1, a_2, a_3, a_4, a_5]$.

GEN mkcol6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v) creates the 6-dimensional column vector [x,y,z,t,u,v] .

GEN mkintmod(GEN x, GEN y) creates the t_INTMOD Mod(x, y). The inputs must be t_INTs satisfying $0 \le x < y$.

GEN mkpolmod(GEN x, GEN y) creates the t_POLMOD Mod(x, y). The input must satisfy $\deg x < \deg y$ with respect to the main variable of the t_POL y. x may be a scalar.

GEN mkmat(GEN x) creates a 1-column t_MAT with column x (a t_COL).

GEN mkmat2(GEN x, GEN y) creates a 2-column t_MAT with columns x, y (t_COLs of the same length).

GEN mkmat22(GEN a, GEN b, GEN c, GEN d) creates the 2 by 2 t_MAT with successive rows $[a, b]$ and $[c, d]$.

GEN mkmat3(GEN x, GEN y, GEN z) creates a 3-column t_MAT with columns x, y, z (t_COLs of the same length).

GEN mkmat4(GEN x, GEN y, GEN z, GEN t) creates a 4-column t_MAT with columns x, y, z, t (t_COLs of the same length).

GEN mkmat5(GEN x, GEN y, GEN z, GEN t, GEN u) creates a 5-column t_MAT with columns x, y, z, t, u (t_COLs of the same length).

GEN mkvec(GEN x) creates a 1-dimensional t_VEC containing x.

GEN mkvec2(GEN x, GEN y) creates a 2-dimensional t_VEC equal to [x,y].

GEN mkvec3(GEN x, GEN y, GEN z) creates a 3-dimensional t_VEC equal to [x,y,z].

GEN mkvec4(GEN x, GEN y, GEN z, GEN t) creates a 4-dimensional t_VEC equal to [x,y,z,t].

GEN mkvec5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5) creates the 5-dimensional t_VEC equal to $[a_1, a_2, a_3, a_4, a_5]$.

GEN mkqfb(GEN a, GEN b, GEN c, GEN D) creates t_QFB equal to Qfb(a,b,c), assuming that $D = b^2 - 4ac$.

GEN mkerr(long n) returns a t_ERROR with error code n (enum err_list).

It is sometimes useful to return such a container whose entries are not universal objects, but nonetheless suitable for gerepileupto. If the entries can be computed at the time the result is returned, the following macros achieve this effect:

GEN retmkvec(GEN x) returns a vector containing the single entry x, where the vector root is created just before the function argument x is evaluated. Expands to

```
{
  GEN res = cgetg(2, t_VEC);
  gel(res, 1) = x; /*  or rather, the expansion of x */
  return res;
}
```

For instance, the retmkvec(gcopy(x)) returns a clean object, just like return mkveccopy(x) would.

GEN retmkvec2(GEN x, GEN y) returns the 2-dimensional t_VEC [x,y].

GEN retmkvec3(GEN x, GEN y, GEN z) returns the 3-dimensional t_VEC [x,y,z].

GEN retmkvec4(GEN x, GEN y, GEN z, GEN t) returns the 4-dimensional t_VEC [x,y,z,t].

GEN retmkvec5(GEN x, GEN y, GEN z, GEN t, GEN u) returns the 5-dimensional row vector [x,y,z,t,u].

GEN retconst_vec(long n, GEN x) returns the n-dimensional t_VEC whose entries are constant and all equal to x.

GEN retmkcol(GEN x) returns the 1-dimensional t_COL [x]~.

GEN retmkcol2(GEN x, GEN y) returns the 2-dimensional t_COL [x,y]~.

GEN retmkcol3(GEN x, GEN y, GEN z) returns the 3-dimensional t_COL [x,y,z]~.

GEN retmkcol4(GEN x, GEN y, GEN z, GEN t) returns the 4-dimensional t_COL [x,y,z,t]~.

GEN retmkcol5(GEN x, GEN y, GEN z, GEN t, GEN u) returns the 5-dimensional column vector [x,y,z,t,u]~.

GEN retmkcol6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v) returns the 6-dimensional column vector [x,y,z,t,u,v]~.

GEN retconst_col(long n, GEN x) returns the n-dimensional t_COL whose entries are constant and all equal to x.

GEN retmkmat(GEN x) returns the 1-column t_MAT with colum x.

GEN retmkmat2(GEN x, GEN y) returns the 2-column t_MAT with columns x, y.

GEN retmkmat3(GEN x, GEN y, GEN z) returns the 3-dimensional t_MAT with columns x, y, z.

GEN retmkmat4(GEN x, GEN y, GEN z, GEN t) returns the 4-dimensional t_MAT with columns x, y, z, t.

GEN retmkmat5(GEN x, GEN y, GEN z, GEN t, GEN u) returns the 5-dimensional t_MAT with columns x, y, z, t, u.

GEN retmkcomplex(GEN x, GEN y) returns the t_COMPLEX x + I*y.

GEN retmkfrac(GEN x, GEN y) returns the t_FRAC x / y. Assume x and y are coprime and $y > 1$.

GEN retmkrfrac(GEN x, GEN y) returns the t_RFRAC x / y. Assume x and y are coprime and more generally that the rational function cannot be simplified.

GEN retmkintmod(GEN x, GEN y) returns the t_INTMOD Mod(x, y).

GEN retmkquad(GEN n, GEN a, GEN b).

GEN retmkpolmod(GEN x, GEN y) returns the t_POLMOD Mod(x, y).

GEN mkintn(long n, ...) returns the nonnegative t_INT whose development in base 2^{32} is given by the following n 32bit-words (unsigned int).

```
mkintn(3, a2, a1, a0);
```

returns $a_2 2^{64} + a_1 2^{32} + a_0$.

GEN mkpoln(long n, ...) Returns the t_POL whose n coefficients (GEN) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial $X^2 + 2X$ (in variable 0, use `setvarn` if you want other variable numbers). Beware that n is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose n coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose n coefficients (`GEN`) follow.

`GEN scalarcol_shallow(GEN x, long n)` creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a shallow copy of `x`. (The name comes from `RgV_isscalar`.)

`GEN scalarmat_shallow(GEN x, long n)` creates an $n \times n$ scalar matrix whose diagonal is set to shallow copies of the scalar `x`.

`GEN RgX_sylvestermatrix(GEN f, GEN g)` return the Sylvester matrix attached to the two `t_POL` in the same variable f and g.

`GEN diagonal_shallow(GEN x)` returns a diagonal matrix whose diagonal is given by the vector x. Shallow function.

`GEN scalarpol_shallow(GEN a, long v)` returns the degree 0 `t_POL` a`pol_x`$(v)^0$.

`GEN deg1pol_shallow(GEN a, GEN b, long v)` returns the degree 1 `t_POL` a`pol_x`$(v) + b$

`GEN deg2pol_shallow(GEN a, GEN b, GEN c, long v)` returns the degree 2 `t_POL` $ax^2 + bx + c$ where $x =$ `pol_x`(v).

`GEN zeropadic_shallow(GEN p, long n)` returns a (shallow) 0 `t_PADIC` equal to $O(\mathtt{p}^{\mathtt{n}})$.

9.3.3 From roots to polynomials.

`GEN deg1_from_roots(GEN L, long v)` given a vector L of scalars, returns the vector of monic linear polynomials in variable v whose roots are the $L[i]$, i.e. the $x - L[i]$.

`GEN roots_from_deg1(GEN L)` given a vector L of monic linear polynomials, return their roots, i.e. the $-L[i](0)$.

`GEN roots_to_pol(GEN L, long v)` given a vector of scalars L, returns the monic polynomial in variable v whose roots are the $L[i]$. Leaves some garbage on stack, but suitable for `gerepileupto`.

`GEN roots_to_pol_r1(GEN L, long v, long r1)` as `roots_to_pol` assuming the first r_1 roots are "real", and the following ones are representatives of conjugate pairs of "complex" roots. So if L has $r_1 + r_2$ elements, we obtain a polynomial of degree $r_1 + 2r_2$. In most applications, the roots are indeed real and complex, but the implementation assumes only that each "complex" root z introduces a quadratic factor $X^2 - \mathtt{trace}(z)X + \mathtt{norm}(z)$. Leaves some garbage on stack, but suitable for `gerepileupto`.

9.4 Integer parts.

`GEN gfloor(GEN x)` creates the floor of `x`, i.e. the (true) integral part.

`GEN gfrac(GEN x)` creates the fractional part of `x`, i.e. `x` minus the floor of `x`.

`GEN gceil(GEN x)` creates the ceiling of `x`.

`GEN ground(GEN x)` rounds towards $+\infty$ the components of `x` to the nearest integers.

`GEN grndtoi(GEN x, long *e)` same as `ground`, but in addition sets `*e` to the binary exponent of $x - \texttt{ground}(x)$. If this is positive, then significant bits are lost in the rounded result. This kind of situation raises an error message in **ground** but not in **grndtoi**. The parameter e can be set to `NULL` if an error estimate is not needed, for a minor speed up.

`GEN gtrunc(GEN x)` truncates `x`. This is the false integer part if `x` is a real number (i.e. the unique integer closest to `x` among those between 0 and `x`). If `x` is a `t_SER`, it is truncated to a `t_POL`; if `x` is a `t_RFRAC`, this takes the polynomial part.

`GEN gtrunc2n(GEN x, long n)` creates the floor of 2^n`x`, this is only implemented for `t_INT`, `t_REAL`, `t_FRAC` and `t_COMPLEX` of those.

`GEN gcvtoi(GEN x, long *e)` analogous to **grndtoi** for `t_REAL` inputs except that rounding is replaced by truncation. Also applies componentwise for vector or matrix inputs; otherwise, sets `*e` to `-HIGHEXPOBIT` (infinite real accuracy) and return `gtrunc(x)`.

9.5 Valuation and shift.

`GEN gshift[z](GEN x, long n[, GEN z])` yields the result of shifting (the components of) `x` left by `n` (if `n` is nonnegative) or right by $-$`n` (if `n` is negative). Applies only to `t_INT` and vectors/matrices of such. For other types, it is simply multiplication by $2^{\texttt{n}}$.

`GEN gmul2n[z](GEN x, long n[, GEN z])` yields the product of `x` and $2^{\texttt{n}}$. This is different from `gshift` when `n` is negative and `x` is a `t_INT`: **gshift** truncates, while **gmul2n** creates a fraction if necessary.

`long gvaluation(GEN x, GEN p)` returns the greatest exponent e such that $\texttt{p}^e$ divides `x`, when this makes sense.

`long gval(GEN x, long v)` returns the highest power of the variable number `v` dividing the `t_POL` `x`.

9.6 Comparison operators.

9.6.1 Generic.

`long gcmp(GEN x, GEN y)` comparison of `x` with `y`: returns 1 ($x > y$), 0 ($x = y$) or -1 ($x < y$). Two `t_STR` are compared using the standard lexicographic ordering; a `t_STR` cannot be compared to any non-string type. If neither x nor y is a `t_STR`, their allowed types are `t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` with positive discriminant (use the canonical embedding $w \to \sqrt{D}/2$ or $w \to (1+\sqrt{D})/2$) or `t_INFINITY`. Use `cmp_universal` to compare arbitrary `GEN`s.

`long lexcmp(GEN x, GEN y)` comparison of `x` with `y` for the lexicographic ordering; when comparing objects of different lengths whose components are all equal up to the smallest of their length, consider that the longest is largest. Consider scalars as 1-component vectors. Return `gcmp`(x, y) if both arguments are scalars.

`int gequalX(GEN x)` return 1 (true) if `x` is a variable (monomial of degree 1 with `t_INT` coefficients equal to 1 and 0), and 0 otherwise

`long gequal(GEN x, GEN y)` returns 1 (true) if `x` is equal to `y`, 0 otherwise. A priori, this makes sense only if `x` and `y` have the same type, in which case they are recursively compared componentwise. When the types are different, a `true` result means that `x - y` was successfully computed and that `gequal0` found it equal to 0. In particular

```
gequal(cgetg(1, t_VEC), gen_0)
```

is true, and the relation is not transitive. E.g. an empty `t_COL` and an empty `t_VEC` are not equal but are both equal to `gen_0`.

`long gidentical(GEN x, GEN y)` returns 1 (true) if `x` is identical to `y`, 0 otherwise. In particular, the types and length of `x` and `y` must be equal. This test is much stricter than `gequal`, in particular, `t_REAL` with different accuracies are tested different. This relation is transitive.

`GEN gmax(GEN x, GEN y)` returns a copy of the maximum of x and y, compared using `gcmp`.

`GEN gmin(GEN x, GEN y)` returns a copy of the minimum of x and y, compared using `gcmp`.

`GEN gmax_shallow(GEN x, GEN y)` shallow version of `gmax`.

`GEN gmin_shallow(GEN x, GEN y)` shallow version of `gmin`.

9.6.2 Comparison with a small integer.

`int isexactzero(GEN x)` returns 1 (true) if `x` is exactly equal to 0 (including `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. This includes recursive objects, for instance vectors, whose components are 0.

`GEN gisexactzero(GEN x)` returns `NULL` unless `x` is exactly equal to 0 (as per `isexactzero`). When `x` is an exact zero return the attached scalar zero as a `t_INT` (`gen_0`), a `t_INTMOD` (`Mod(0,`N`)` for the largest possible N) or a `t_FFELT`.

`int isrationalzero(GEN x)` returns 1 (true) if `x` is equal to an integer 0 (excluding `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. Contrary to `isintzero`, this includes recursive objects, for instance vectors, whose components are 0.

`int ismpzero(GEN x)` returns 1 (true) if `x` is a `t_INT` or a `t_REAL` equal to 0.

`int isintzero(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 0.

`int isint1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 1.

`int isintm1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to -1.

`int equali1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to 1, and return 0 (false) otherwise.

`int equalim1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to -1, and return 0 (false) otherwise.

`int is_pm1(GEN x)`. Assuming that `x` is a *nonzero* `t_INT`, return 1 (true) if `x` is equal to -1 or 1, and return 0 (false) otherwise.

`int gequal0(GEN x)` returns 1 (true) if `x` is equal to 0, 0 (false) otherwise.

`int gequal1(GEN x)` returns 1 (true) if `x` is equal to 1, 0 (false) otherwise.

`int gequalm1(GEN x)` returns 1 (true) if `x` is equal to -1, 0 (false) otherwise.

`long gcmpsg(long s, GEN x)`

`long gcmpgs(GEN x, long s)` comparison of `x` with the `long s`.

`GEN gmaxsg(long s, GEN x)`

`GEN gmaxgs(GEN x, long s)` returns the largest of `x` and the `long s` (converted to `GEN`)

`GEN gminsg(long s, GEN x)`

`GEN gmings(GEN x, long s)` returns the smallest of `x` and the `long s` (converted to `GEN`)

`long gequalsg(long s, GEN x)`

`long gequalgs(GEN x, long s)` returns 1 (true) if `x` is equal to the `long s`, 0 otherwise.

9.7 Miscellaneous Boolean functions.

`int isrationalzeroscalar(GEN x)` equivalent to, but faster than,

```
is_scalar_t(typ(x)) && isrationalzero(x)
```

`int isinexact(GEN x)` returns 1 (true) if x has an inexact component, and 0 (false) otherwise.

`int isinexactreal(GEN x)` return 1 if x has an inexact `t_REAL` component, and 0 otherwise.

`int isrealappr(GEN x, long e)` applies (recursively) to complex inputs; returns 1 if x is approximately real to the bit accuracy e, and 0 otherwise. This means that any `t_COMPLEX` component must have imaginary part t satisfying `gexpo`$(t) < e$.

`int isint(GEN x, GEN *n)` returns 0 (false) if `x` does not round to an integer. Otherwise, returns 1 (true) and set `n` to the rounded value.

`int issmall(GEN x, long *n)` returns 0 (false) if `x` does not round to a small integer (suitable for `itos`). Otherwise, returns 1 (true) and set `n` to the rounded value.

`long iscomplex(GEN x)` returns 1 (true) if `x` is a complex number (of component types embeddable into the reals) but is not itself real, 0 if `x` is a real (not necessarily of type `t_REAL`), or raises an error if `x` is not embeddable into the complex numbers.

9.7.1 Obsolete.

The following less convenient comparison functions and Boolean operators were used by the historical GP interpreter. They are provided for backward compatibility only and should not be used:

`GEN gle(GEN x, GEN y)`

`GEN glt(GEN x, GEN y)`

`GEN gge(GEN x, GEN y)`

`GEN ggt(GEN x, GEN y)`

`GEN geq(GEN x, GEN y)`

`GEN gne(GEN x, GEN y)`

`GEN gor(GEN x, GEN y)`

`GEN gand(GEN x, GEN y)`

`GEN gnot(GEN x, GEN y)`

9.8 Sorting.

9.8.1 Basic sort.

`GEN sort(GEN x)` sorts the vector `x` in ascending order using a mergesort algorithm, and `gcmp` as the underlying comparison routine (returns the sorted vector). This routine copies all components of x, use `gen_sort_inplace` for a more memory-efficient function.

`GEN lexsort(GEN x)`, as `sort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN vecsort(GEN x, GEN k)`, as `sort`, but sorts the vector `x` in ascending *lexicographic* order, according to the entries of the `t_VECSMALL` `k`. For example, if $\texttt{k} = [2, 1, 3]$, sorting will be done with respect to the second component, and when these are equal, with respect to the first, and when these are equal, with respect to the third.

9.8.2 Indirect sorting.

`GEN indexsort(GEN x)` as `sort`, but only returns the permutation which, applied to `x`, would sort the vector. The result is a `t_VECSMALL`.

`GEN indexlexsort(GEN x)`, as `indexsort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN indexvecsort(GEN x, GEN k)`, as `vecsort`, but only returns the permutation that would sort the vector `x`.

`long vecindexmin(GEN x)` returns the index for a minimal element of x (`t_VEC`, `t_COL` or `t_VECSMALL`).

`long vecindexmax(GEN x)` returns the index for a maximal element of x (`t_VEC`, `t_COL` or `t_VECSMALL`).

9.8.3 Generic sort and search. The following routines allow to use an arbitrary comparison function `int (*cmp)(void* data, GEN x, GEN y)`, such that `cmp(data,x,y)` returns a negative result if $x < y$, a positive one if $x > y$ and 0 if $x = y$. The `data` argument is there in case your `cmp` requires additional context.

`GEN gen_sort(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `sort`, with an explicit comparison routine.

`GEN gen_sort_shallow(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, shallow variant of `gen_sort`.

`GEN gen_sort_uniq(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `gen_sort`, removing duplicate entries.

`GEN gen_indexsort(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`.

`GEN gen_indexsort_uniq(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`, removing duplicate entries.

`void gen_sort_inplace(GEN x, void *data, int (*cmp)(void*, GEN, GEN), GEN *perm)` sort `x` in place, without copying its components. If `perm` is not `NULL`, it is set to the permutation that would sort the original `x`.

`GEN gen_setminus(GEN A, GEN B, int (*cmp)(GEN, GEN))` given two sorted vectors A and B, returns the vector of elements of A not belonging to B.

`GEN sort_factor(GEN y, void *data, int (*cmp)(void *, GEN, GEN))`: assuming `y` is a factorization matrix, sorts its rows in place (no copy is made) according to the comparison function `cmp` applied to its first column.

`GEN merge_sort_uniq(GEN x, GEN y, void *data, int (*cmp)(void *, GEN, GEN))` assuming `x` and `y` are sorted vectors, with respect to the `cmp` comparison function, return a sorted concatenation, with duplicates removed. Shallow function.

`GEN setunion_i(GEN x, GEN y)` shallow version of `setunion`, a simple alias for

```
merge_sort_uniq(x,y, (void*)cmp_universal, cmp_nodata)
```

`GEN merge_factor(GEN fx, GEN fy, void *data, int (*cmp)(void *, GEN, GEN))` let `fx` and `fy` be factorization matrices for X and Y sorted with respect to the comparison function `cmp` (see `sort_factor`), returns the factorization of $X * Y$.

`long gen_search(GEN v, GEN y, void *data, int (*cmp)(void*, GEN, GEN))`.
Let v be a vector sorted according to `cmp(data,a,b)`; look for an index i such that $v[i]$ is equal to y. If y is found, return i (not necessarily the first occurence in case of multisets), else return $-i$ where i is the index where y should be inserted.

`long tablesearch(GEN T, GEN x, int (*cmp)(GEN, GEN))` is a faster implementation for the common case `gen_search(T,x,cmp,cmp_nodata)` when we have no need to insert missing elements; return 0 in case x is not found.

9.8.4 Further useful comparison functions.

`int cmp_universal(GEN x, GEN y)` a somewhat arbitrary universal comparison function, devoid of sensible mathematical meaning. It is transitive, and returns 0 if and only if `gidentical(x,y)` is true. Useful to sort and search vectors of arbitrary data.

`int cmp_nodata(void *data, GEN x, GEN y)`. This function is a hack used to pass an existing basic comparison function lacking the `data` argument, i.e. with prototype `int (*cmp)(GEN x, GEN y)`. Instead of `gen_sort(x, NULL, cmp)` which may or may not work depending on how your compiler handles typecasts between incompatible function pointers, one should use `gen_sort(x, (void*)cmp, cmp_nodata)`.

Here are a few basic comparison functions, to be used with `cmp_nodata`:

`int ZV_cmp(GEN x, GEN y)` compare two `ZV`, which we assume have the same length (lexicographic order).

`int cmp_Flx(GEN x, GEN y)` compare two `Flx`, which we assume have the same main variable (lexicographic order).

`int cmp_RgX(GEN x, GEN y)` compare two polynomials, which we assume have the same main variable (lexicographic order). The coefficients are compared using `gcmp`.

`int cmp_prime_over_p(GEN x, GEN y)` compare two prime ideals, which we assume divide the same prime number. The comparison is ad hoc but orders according to increasing residue degrees.

`int cmp_prime_ideal(GEN x, GEN y)` compare two prime ideals in the same nf. Orders by increasing primes, breaking ties using `cmp_prime_over_p`.

`int cmp_padic(GEN x, GEN y)` compare two `t_PADIC` (for the same prime p).

Finally a more elaborate comparison function:

`int gen_cmp_RgX(void *data, GEN x, GEN y)` compare two polynomials, ordering first by increasing degree, then according to the coefficient comparison function:

```
int (*cmp_coeff)(GEN,GEN) = (int(*)(GEN,GEN)) data;
```

9.9 Division.

`GEN gdivgu(GEN x, ulong u)` return x/u).

`GEN gdivgunextu(GEN x, ulong u)` return $x/(u(u+1))$. If $u(u+1)$ does not fit into an `ulong`, it is created and left on the stack for efficiency.

`GEN divrunextu(GEN x, ulong i)` as `gdivgunextu` for a `t_REAL` x.

9.10 Divisibility, Euclidean division.

GEN gdivexact(GEN x, GEN y) returns the quotient x/y, assuming y divides x. Not stack clean if $y = 1$ (we return x, not a copy).

int gdvd(GEN x, GEN y) returns 1 (true) if y divides x, 0 otherwise.

GEN gdiventres(GEN x, GEN y) creates a 2-component vertical vector whose components are the true Euclidean quotient and remainder of x and y.

GEN gdivent[z](GEN x, GEN y[, GEN z]) yields the true Euclidean quotient of x and the t_INT or t_POL y, as per the \ GP operator.

GEN gdiventsg(long s, GEN y[, GEN z]), as gdivent except that x is a long.

GEN gdiventgs[z](GEN x, long s[, GEN z]), as gdivent except that y is a long.

GEN gmod[z](GEN x, GEN y[, GEN z]) yields the remainder of x modulo the t_INT or t_POL y, as per the % GP operator. A t_REAL or t_FRAC y is also allowed, in which case the remainder is the unique real r such that $0 \leq r < |\mathrm{y}|$ and $\mathrm{y} = q\mathrm{x} + r$ for some (in fact unique) integer q.

GEN gmodsg(long s, GEN y[, GEN z]) as gmod, except x is a long.

GEN gmodgs(GEN x, long s[, GEN z]) as gmod, except y is a long.

GEN gdivmod(GEN x, GEN y, GEN *r) If r is not equal to NULL or ONLY_REM, creates the (false) Euclidean quotient of x and y, and puts (the address of) the remainder into *r. If r is equal to NULL, do not create the remainder, and if r is equal to ONLY_REM, create and output only the remainder. The remainder is created after the quotient and can be disposed of individually with a cgiv(r).

GEN poldivrem(GEN x, GEN y, GEN *r) same as **gdivmod** but specifically for t_POLs x and y, not necessarily in the same variable. Either of x and y may also be scalars, treated as polynomials of degree 0.

GEN gdeuc(GEN x, GEN y) creates the Euclidean quotient of the t_POLs x and y. Either of x and y may also be scalars, treated as polynomials of degree 0.

GEN grem(GEN x, GEN y) creates the Euclidean remainder of the t_POL x divided by the t_POL y. Either of x and y may also be scalars, treated as polynomials of degree 0.

GEN gdivround(GEN x, GEN y) if x and y are real (t_INT, t_REAL, t_FRAC), return the rounded Euclidean quotient of x and y as per the \/ GP operator. Operate componentwise if x is a t_COL, t_VEC or t_MAT. Otherwise as **gdivent**.

GEN centermod_i(GEN x, GEN y, GEN y2), as centermodii, componentwise.

GEN centermod(GEN x, GEN y), as centermod_i, except that y2 is computed (and left on the stack for efficiency).

GEN ginvmod(GEN x, GEN y) creates the inverse of x modulo y when it exists. y must be of type t_INT (in which case x is of type t_INT) or t_POL (in which case x is either a scalar type or a t_POL).

9.11 GCD, content and primitive part.

9.11.1 Generic.

GEN resultant(GEN x, GEN y) creates the resultant of the t_POLs x and y computed using Sylvester's matrix (inexact inputs), a modular algorithm (inputs in $\mathbf{Q}[X]$) or the subresultant algorithm, as optimized by Lazard and Ducos. Either of x and y may also be scalars (treated as polynomials of degree 0)

GEN ggcd(GEN x, GEN y) creates the GCD of x and y.

GEN glcm(GEN x, GEN y) creates the LCM of x and y.

GEN gbezout(GEN x, GEN y, GEN *u, GEN *v) returns the GCD of x and y, and puts (the addresses of) objects u and v such that $u\mathtt{x} + v\mathtt{y} = \gcd(\mathtt{x}, \mathtt{y})$ into *u and *v.

GEN subresext(GEN x, GEN y, GEN *U, GEN *V) returns the resultant of x and y, and puts (the addresses of) polynomials u and v such that $u\mathtt{x} + v\mathtt{y} = \mathrm{Res}(\mathtt{x}, \mathtt{y})$ into *U and *V.

GEN content(GEN x) returns the GCD of all the components of x.

GEN primitive_part(GEN x, GEN *c) sets c to content(x) and returns the primitive part x / c. A trivial content is set to NULL.

GEN primpart(GEN x) as above but the content is lost. (For efficiency, the content remains on the stack.)

GEN denom_i(GEN x) shallow version of denom.

GEN numer_i(GEN x) shallow version of numer.

9.11.2 Over the rationals.

long Q_pval(GEN x, GEN p) valuation at the t_INT p of the t_INT or t_FRAC x.

long Q_lval(GEN x, ulong p) same for ulong p.

long Q_pvalrem(GEN x, GEN p, GEN *r) returns the valuation e at the t_INT p of the t_INT or t_FRAC x. The quotient $\mathtt{x}/\mathtt{p}^e$ is returned in *r.

long Q_lvalrem(GEN x, ulong p, GEN *r) same for ulong p.

GEN Q_abs(GEN x) absolute value of the t_INT or t_FRAC x.

GEN Qdivii(GEN x, GEN y), assuming x and y are both of type t_INT, return the quotient x/y as a t_INT or t_FRAC; marginally faster than gdiv.

GEN Qdivis(GEN x, long y), assuming x is an t_INT, return the quotient x/y as a t_INT or t_FRAC; marginally faster than gdiv.

GEN Qdiviu(GEN x, ulong y), assuming x is an t_INT, return the quotient x/y as a t_INT or t_FRAC; marginally faster than gdiv.

GEN Q_abs_shallow(GEN x) x being a t_INT or a t_FRAC, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and gneg(x) otherwise.

GEN Q_gcd(GEN x, GEN y) gcd of the t_INT or t_FRAC x and y.

In the following functions, arguments belong to a $M \otimes_{\mathbf{Z}} \mathbf{Q}$ for some natural $\mathbf{Z}$-module M, e.g. multivariate polynomials with integer coefficients (or vectors/matrices recursively built from such objects), and an element of M is said to be *integral*. We are interested in contents, denominators, etc. with respect to this canonical integral structure; in particular, contents belong to $\mathbf{Q}$, denominators to $\mathbf{Z}$. For instance the $\mathbf{Q}$-content of $(1/2)xy$ is $(1/2)$, and its $\mathbf{Q}$-denominator is 2, whereas `content` would return $y/2$ and `denom` 1.

`GEN Q_content(GEN x)` the $\mathbf{Q}$-content of x.

`GEN Z_content(GEN x)` as `Q_content` but assume that all rationals are in fact `t_INT`s and return `NULL` when the content is 1. This function returns as soon as the content is found to equal 1.

`GEN Q_content_safe(GEN x)` as `Q_content`, returning `NULL` when the $\mathbf{Q}$-content is not defined (e.g. for a `t_REAL` or `t_INTMOD` component).

`GEN Q_denom(GEN x)` the $\mathbf{Q}$-denominator of x. Shallow function. Raises en `e_TYPE` error out when the notion is meaningless, e.g. for a `t_REAL` or `t_INTMOD` component.

`GEN Q_denom_safe(GEN x)` the $\mathbf{Q}$-denominator of x. Shallow function. Return `NULL` when the notion is meaningless.

`GEN Q_primitive_part(GEN x, GEN *c)` sets `c` to the $\mathbf{Q}$-content of `x` and returns `x / c`, which is integral.

`GEN Q_primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

`GEN vec_Q_primpart(GEN x)` as above component-wise. Applied to a `t_MAT`, the result has primitive columns.

`GEN row_Q_primpart(GEN x)` as above, applied to the rows of a `t_MAT`, so that the result has primitive rows. Not `gerepile`-safe.

`GEN Q_remove_denom(GEN x, GEN *ptd)` sets `d` to the $\mathbf{Q}$-denominator of `x` and returns `x * d`, which is integral. Shallow function.

`GEN Q_div_to_int(GEN x, GEN c)` returns `x / c`, assuming c is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

`GEN Q_mul_to_int(GEN x, GEN c)` returns `x * c`, assuming c is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

`GEN Q_muli_to_int(GEN x, GEN d)` returns `x * c`, assuming c is a `t_INT` and the result is integral.

`GEN mul_content(GEN cx, GEN cy)` `cx` and `cy` are as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns their product (either a `GEN` or `NULL`).

`GEN div_content(GEN cx, GEN cy)` `cx` and `cy` are as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns their quotient (either a `GEN` or `NULL`).

`GEN inv_content(GEN c)` c is as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns its inverse (either a `GEN` or `NULL`).

`GEN mul_denom(GEN dx, GEN dy)` `dx` and `dy` are as set by `Q_remove_denom`: either a `t_INT` or `NULL` representing the trivial denominator 1. Returns their product (either a `t_INT` or `NULL`).

9.12 Generic arithmetic operators.

9.12.1 Unary operators.

GEN gneg[z](GEN x[, GEN z]) yields $-$x.

GEN gneg_i(GEN x) shallow function yielding $-$x.

GEN gabs[z](GEN x[, GEN z]) yields |x|.

GEN gsqr(GEN x) creates the square of x.

GEN ginv(GEN x) creates the inverse of x.

9.12.2 Binary operators.

Let "*op*" be a binary operation among

op=**add**: addition (x + y).

op=**sub**: subtraction (x - y).

op=**mul**: multiplication (x * y).

op=**div**: division (x / y).

The names and prototypes of the functions corresponding to *op* are as follows:

GEN g*op*(GEN x, GEN y)

GEN g*op*gs(GEN x, long s)

GEN g*op*gu(GEN x, ulong u)

GEN g*op*sg(long s, GEN y)

GEN g*op*ug(ulong u, GEN y)

Explicitly

GEN gadd(GEN x, GEN y), GEN gaddgs(GEN x, long s), GEN gaddsg(long s, GEN x)

GEN gmul(GEN x, GEN y), GEN gmulgs(GEN x, long s), GEN gmulsg(long s, GEN x), GEN gmulgu(GEN x, ulong u), GEN gmulug(GEN x, ulong u),

GEN gsub(GEN x, GEN y), GEN gsubgs(GEN x, long s), GEN gsubsg(long s, GEN x)

GEN gdiv(GEN x, GEN y), GEN gdivgs(GEN x, long s), GEN gdivsg(long s, GEN x), GEN gdivgu(GEN x, ulong u),

GEN gpow(GEN x, GEN y, long l) creates $\mathtt{x}^{\mathtt{y}}$. If y is a t_INT, return powgi(x,y) (the precision l is not taken into account). Otherwise, the result is $\exp(\mathtt{y} * \log(\mathtt{x}))$ where exact arguments are converted to floats of precision l in case of need; if there is no need, for instance if x is a t_REAL, l is ignored. Indeed, if x is a t_REAL, the accuracy of $\log x$ is determined from the accuracy of x, it is no problem to multiply by y, even if it is an exact type, and the accuracy of the exponential is determined, exactly as in the case of the initial $\log x$.

GEN gpowgs(GEN x, long n) creates $\mathtt{x}^{\mathtt{n}}$ using binary powering. To treat the special case $n = 0$, we consider gpowgs as a series of gmul, so we follow the rule of returning result which is as exact as possible given the input. More precisely, we return

• gen_1 if x has type t_INT, t_REAL, t_FRAC, or t_PADIC

• Mod(1,N) if x is a t_INTMOD modulo N.

• gen_1 for t_COMPLEX, t_QUAD unless one component is a t_INTMOD, in which case we return Mod(1, N) for a suitable N (the gcd of the moduli that appear).

• FF_1(x) for a t_FFELT.

• qfb_1(x) for t_QFB.

• the identity permutation for t_VECSMALL.

• Rg_get_1(x) otherwise

Of course, the only practical use of this routine for $n = 0$ is to obtain the multiplicative neutral element in the base ring (or to treat marginal cases that should be special cased anyway if there is the slightest doubt about what the result should be).

GEN powgi(GEN x, GEN y) creates x^y, where y is a t_INT, using left-shift binary powering. The case where $y = 0$ (as all cases where y is small) is handled by gpowgs(x, 0).

GEN gpowers(GEN x, long n) returns the vector $[1, x, \ldots, x^n]$.

GEN grootsof1(long n, long prec) returns the vector $[1, x, \ldots, x^{n-1}]$, where x is the n-th root of unity $\exp(2i\pi/n)$.

GEN gsqrpowers(GEN x, long n) returns the vector $[x, x^4, \ldots, x^{n^2}]$.

In addition we also have the obsolete forms:

void gaddz(GEN x, GEN y, GEN z)

void gsubz(GEN x, GEN y, GEN z)

void gmulz(GEN x, GEN y, GEN z)

void gdivz(GEN x, GEN y, GEN z)

9.13 Generic operators: product, powering, factorback.

To describe the following functions, we use the following private typedefs to simplify the description:

```
typedef (*F0)(void *);
typedef (*F1)(void *, GEN);
typedef (*F2)(void *, GEN, GEN);
```

They correspond to generic functions with one and two arguments respectively (the void* argument provides some arbitrary evaluation context).

GEN gen_product(GEN v, void *D, F2 op) Given two objects x, y, assume that op(D, x, y) implements an associative binary operator. If v has k entries, return

$$v[1] \ op \ v[2] \ op \ \ldots \ op \ v[k];$$

returns gen_1 if $k = 0$ and a copy of $v[1]$ if $k = 1$. Use divide and conquer strategy. Leave some garbage on stack, but suitable for gerepileupto if mul is.

`GEN gen_pow(GEN x, GEN n, void *D, F1 sqr, F2 mul)` $n > 0$ a `t_INT`, returns x^n; `mul(D,` x`,` y`)` implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul(D,` x`,` x`)`.

`GEN gen_powu(GEN x, ulong n, void *D, F1 sqr, F2 mul)` $n > 0$, returns x^n. See `gen_pow`.

`GEN gen_pow_i(GEN x, GEN n, void *E, F1 sqr, F2 mul)` internal variant of `gen_pow`, not memory-clean.

`GEN gen_powu_i(GEN x, ulong n, void *E, F1 sqr, F2 mul)` internal variant of `gen_powu`, not memory-clean.

`GEN gen_pow_fold(GEN x, GEN n, void *D, F1 sqr, F1 msqr)` variant of `gen_pow`, where `mul` is replaced by `msqr`, with `msqr(D,` y`)` returning xy^2. In particular `D` must implicitly contain x.

`GEN gen_pow_fold_i(GEN x, GEN n, void *E, F1 sqr, F1 msqr)` internal variant of the function `gen_pow_fold`, not memory-clean.

`GEN gen_powu_fold(GEN x, ulong n, void *D, F1 sqr, F1 msqr)`, see `gen_pow_fold`.

`GEN gen_powu_fold_i(GEN x, ulong n, void *E, F1 sqr, F1 msqr)` see `gen_pow_fold_i`.

`GEN gen_pow_init(GEN x, GEN n, long k, void *E, GEN (*sqr)(void*, GEN), GEN (*mul)(void*, GEN, GEN))` Return a table `R` that can be used with `gen_pow_table` to compute the powers of x up to n. The table is of size $2^k \log_2(n)$.

`GEN gen_pow_table(GEN R, GEN n, void *E, GEN (*one)(void*), GEN (*mul)(void*, GEN, GEN))`

Return x^n, where R is as given by `gen_pow_init(x,m,k,E,sqr,mul)` for some integer $m \geq n$.

`GEN gen_powers(GEN x, long n, long usesqr, void *D, F1 sqr, F2 mul, F0 one)` returns $[\mathtt{x}^0, \ldots, \mathtt{x}^\mathtt{n}]$ as a `t_VEC`; `mul(D,` x`,` y`)` implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul(D,` x`,` x`)`; `one` returns the monoid unit. The flag `usesqr` should be set to 1 if squaring are faster than multiplication by x.

`GEN gen_factorback(GEN L, GEN e, void *D, F2 mul, F2 pow, GEN (*one)(void *)D)` generic form of `factorback`. The pair $[L, e]$ is of the form

• `[fa, NULL]`, `fa` a two-column factorization matrix: expand it.

• `[v, NULL]`, v a vector of objects: return their product.

• or `[v, e]`, v a vector of objects, e a vector of integral exponents (a `ZV` or `zv`): return the product of the $v[i]^{e[i]}$.

`mul(D,` x`,` y`)` and `pow(D,` x`,` n`)` return xy and x^n respectively.

`one(D)` returns the neutral element. If `one` is `NULL`, `gen_1` is used instead.

9.14 Matrix and polynomial norms.

This section concerns only standard norms of **R** and **C** vector spaces, not algebraic norms given by the determinant of some multiplication operator. We have already seen type-specific functions like `ZM_supnorm` or `RgM_fpnorml2` and limit ourselves to generic functions assuming nothing about their `GEN` argument; these functions allow the following scalar types: `t_INT`, `t_FRAC`, `t_REAL`, `t_COMPLEX`, `t_QUAD` and are defined recursively (in terms of norms of their components) for the following "container" types: `t_POL`, `t_VEC`, `t_COL` and `t_MAT`. They raise an error if some other type appears in the argument.

`GEN gnorml2(GEN x)` The norm of a scalar is the square of its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the *square* of the usual L^2 norm. In most applications, the missing square root computation can be skipped.

`GEN gnorml1(GEN x, long prec)` The norm of a scalar is its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the usual L^1 norm. One must include a real precision `prec` in case the inputs include `t_COMPLEX` or `t_QUAD` with exact rational components: a square root must be computed and we must choose an accuracy.

`GEN gnorml1_fake(GEN x)` as `gnorml1`, except that the norm of a `t_QUAD` $x + wy$ or `t_COMPLEX` $x + Iy$ is defined as $|x| + |y|$, where we use the ordinary real absolute value. This is still a norm of **R** vector spaces, which is easier to compute than `gnorml1` and can often be used in its place.

`GEN gsupnorm(GEN x, long prec)` The norm of a scalar is its complex modulus, the norm of a recursive type is the max of the norms of its components. A precision `prec` must be included for the same reason as in `gnorml1`.

`void gsupnorm_aux(GEN x, GEN *m, GEN *m2, long prec)` is the low-level function underlying `gsupnorm`, used as follows:

```
GEN m = NULL, m2 = NULL;
gsupnorm_aux(x, &m, &m2);
```

After the call, the sup norm of x is the min of `m` and the square root of `m2`; one or both of `m`, `m2` may be `NULL`, in which case it must be omitted. You may initially set `m` and `m2` to non-`NULL` values, in which case, the above procedure yields the max of (the initial) `m`, the square root of (the initial) `m2`, and the sup norm of x.

The strange interface is due to the fact that $|z|^2$ is easier to compute than $|z|$ for a `t_QUAD` or `t_COMPLEX` z: `m2` is the max of those $|z|^2$, and `m` is the max of the other $|z|$.

9.15 Substitution and evaluation.

`GEN gsubst(GEN x, long v, GEN y)` substitutes the object `y` into `x` for the variable number `v`.

`GEN poleval(GEN q, GEN x)` evaluates the `t_POL` or `t_RFRAC` q at x. For convenience, a `t_VEC` or `t_COL` is also recognized as the `t_POL` `gtovecrev(q)`.

`GEN RgX_cxeval(GEN T, GEN x, GEN xi)` evaluate the `t_POL` T at x via Horner's scheme. If xi is not `NULL` it must be equal to $1/x$ and we evaluate $x^{\deg T}T(1/x)$ instead. This is useful when $|x| > 1$ is a `t_REAL` or an inexact `t_COMPLEX` and T has "balanced" coefficients, since the evaluation becomes numerically stable.

`GEN RgXY_cxevalx(GEN T, GEN x, GEN xi)` Apply `RgX_cxeval` to all the polynomials coefficients of T.

`GEN RgX_RgM_eval(GEN q, GEN x)` evaluates the `t_POL` q at the square matrix x.

`GEN RgX_RgMV_eval(GEN f, GEN V)` returns the evaluation `f`(`x`), assuming that `V` was computed by `FpXQ_powers`(`x`, n) for some $n > 1$.

`GEN qfeval(GEN q, GEN x)` evaluates the quadratic form q (symmetric matrix) at x (column vector of compatible dimensions).

`GEN qfevalb(GEN q, GEN x, GEN y)` evaluates the polar bilinear form attached to the quadratic form q (symmetric matrix) at x, y (column vectors of compatible dimensions).

`GEN hqfeval(GEN q, GEN x)` evaluates the Hermitian form q (a Hermitian complex matrix) at x.

`GEN qf_apply_RgM(GEN q, GEN M)` q is a symmetric $n \times n$ matrix, M an $n \times k$ matrix, return $M'qM$.

`GEN qf_apply_ZM(GEN q, GEN M)` as above assuming that both q and M have integer entries.

Chapter 10:
Miscellaneous mathematical functions

10.1 Fractions.

`GEN absfrac(GEN x)` returns the absolute value of the `t_FRAC` x.

`GEN absfrac_shallow(GEN x)` x being a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg`(x) otherwise.

`GEN sqrfrac(GEN x)` returns the square of the `t_FRAC` x.

10.2 Binomials.

`GEN binomial(GEN x, long k)`

`GEN binomialuu(ulong n, ulong k)`

`GEN vecbinomial(long n)`, which returns a vector v with $n + 1$ `t_INT` components such that $v[k + 1] =$ `binomial`(n, k) for k from 0 up to n.

10.3 Real numbers.

`GEN R_abs(GEN x)` x being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns $|x|$.

`GEN R_abs_shallow(GEN x)` x being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg`(x) otherwise.

`GEN modRr_safe(GEN x, GEN y)` let x be a `t_INT`, a `t_REAL` or `t_FRAC` and let y be a `t_REAL`. Return $x\%y$ unless the input accuracy is unsufficient to compute the floor or x/y in which case we return `NULL`.

10.4 Complex numbers.

GEN gimag(GEN x) returns a copy of the imaginary part of x.

GEN greal(GEN x) returns a copy of the real part of x. If x is a t_QUAD, returns the coefficient of 1 in the "canonical" integral basis $(1, \omega)$.

GEN gconj(GEN x) returns greal(x) − 2gimag(x), which is the ordinary complex conjugate except for a real t_QUAD.

GEN imag_i(GEN x), shallow variant of gimag.

GEN real_i(GEN x), shallow variant of greal.

GEN conj_i(GEN x), shallow variant of gconj.

GEN mulreal(GEN x, GEN y) returns the real part of xy; x, y have type t_INT, t_FRAC, t_REAL or t_COMPLEX. See also RgM_mulreal.

GEN cxnorm(GEN x) norm of the t_COMPLEX x (modulus squared).

GEN cxexpm1(GEN x) returns $\exp(x) - 1$, for a t_COMPLEX x.

int cx_approx_equal(GEN a, GEN b) test whether (t_INT, t_FRAC, t_REAL, or t_COMPLEX of those) a and b are approximately equal. This returns 1 if and only if the division by $a - b$ would produce a division by 0 (which is a less stringent test than testing whether $a - b$ evaluates to 0).

int cx_approx0(GEN a, GEN b) test whether (t_INT, t_FRAC, t_REAL, or t_COMPLEX of those) a is approximately 0, where b is a reference point. A non-0 t_REAL component x is approximately 0 if

$$\texttt{exponent}(b) - \texttt{exponent}(x) > \texttt{bit\_prec(x)}\ .$$

10.5 Quadratic numbers and binary quadratic forms.

GEN quad_disc(GEN x) returns the discriminant of the t_QUAD x. Not stack-clean but suitable for gerepileupto.

GEN quadnorm(GEN x) norm of the t_QUAD x.

GEN qfb_disc(GEN x) returns the discriminant of the t_QFB x.

GEN qfb_disc3(GEN x, GEN y, GEN z) returns $y^2 - 4xz$ assuming all inputs are t_INTs. Not stack-clean.

GEN qfb_apply_ZM(GEN q, GEN g) returns $q \circ g$.

GEN qfbforms(GEN D) given a discriminant $D < 0$, return the list of reduced forms of discriminant D as t_VECSMALL with 3 components. The primitive forms in the list enumerate the class group of the quadratic order of discriminant D; if D is fundamental, all returned forms are automatically primitive.

10.6 Polynomials.

`GEN truecoef(GEN x, long n)` returns `polcoef(x,n, -1)`, i.e. the coefficient of the term of degree `n` in the main variable. This is a safe but expensive function that must *copy* its return value so that it be `gerepile`-safe. Use `polcoef_i` for a fast internal variant.

`GEN polcoef_i(GEN x, long n, long v)` internal shallow function. Rewrite x as a Laurent polynomial in the variable v and returns its coefficient of degree n (`gen_0` if this falls outside the coefficient array). Allow `t_POL`, `t_SER`, `t_RFRAC` and scalars.

`long degree(GEN x)` returns `poldegree(x, -1)`, the degree of `x` with respect to its main variable, with the usual meaning if the leading coefficient of x is nonzero. If the sign of x is 0, this function always returns -1. Otherwise, we return the index of the leading coefficient of x, i.e. the coefficient of largest index stored in x. For instance the "degrees" of

```
0. E-38 * x^4 + 0.E-19 * x + 1
Mod(0,2) * x^0    \\ sign is 0 !
```

are 4 and -1 respectively.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL x` with respect to its main variable, *if* its leading coefficient is nonzero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. `Mod(0,2)`). If x has no coefficients (rational 0 polynomial), its length is 2 and we return the expected -1.

`GEN characteristic(GEN x)` returns the characteristic of the base ring over which the polynomial is defined (as defined by `t_INTMOD` and `t_FFELT` components). The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

`GEN residual_characteristic(GEN x)` returns a kind of "residual characteristic" of the base ring over which the polynomial is defined. This is defined as the gcd of all moduli `t_INTMOD`s occurring in the structure, as well as primes p arising from `t_PADIC`s or `t_FFELT`s. The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

`GEN resultant(GEN x, GEN y)` resultant of `x` and `y`, with respect to the main variable of highest priority. Uses either the subresultant algorithm (generic case), a modular algorithm (inputs in $\mathbf{Q}[X]$) or Sylvester's matrix (inexact inputs).

`GEN resultant2(GEN x, GEN y)` resultant of `x` and `y`, with respect to the main variable of highest priority. Computes the determinant of Sylvester's matrix.

`GEN cleanroots(GEN x, long prec)` returns the complex roots of the complex polynomial x (with coefficients `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX` of the above). The roots are returned as `t_REAL` or `t_COMPLEX` of `t_REAL`s of precision `prec` (guaranteeing a nonzero imaginary part). See `QX_complex_roots`.

`double fujiwara_bound(GEN x)` return a quick upper bound for the logarithm in base 2 of the modulus of the largest complex roots of the polynomial x (complex coefficients).

`double fujiwara_bound_real(GEN x, long sign)` return a quick upper bound for the logarithm in base 2 of the absolute value of the largest real root of sign *sign* (1 or -1), for the polynomial x (real coefficients).

`GEN polmod_to_embed(GEN x, long prec)` return the vector of complex embeddings of the `t_POLMOD` x (with complex coefficients). Shallow function, simple complex variant of `conjvec`.

`GEN pollegendre_reduced(long n, long v)` let $P_n(t) \in \mathbf{Q}[t]$ be the n-th Legendre polynomial in variable v. Return $p \in \mathbf{Z}[t]$ such that $2^n P_n(t) = p(t^2)$ (n even) or $tp(t^2)$ (n odd).

10.7 Power series.

`GEN sertoser(GEN x, long prec)` return the `t_SER` x truncated or extended (with zeros) to `prec` terms. Shallow function, assume that `prec` ≥ 0.

`GEN derivser(GEN x)` returns the derivative of the `t_SER` `x` with respect to its main variable.

`GEN integser(GEN x)` returns the primitive of the `t_SER` `x` with respect to its main variable.

`GEN truecoef(GEN x, long n)` returns `polcoef(x,n, -1)`, i.e. the coefficient of the term of degree `n` in the main variable. This is a safe but expensive function that must *copy* its return value so that it be `gerepile`-safe. Use `polcoef_i` for a fast internal variant.

`GEN ser_unscale(GEN P, GEN h)` return $P(hx)$, not memory clean.

`GEN ser_normalize(GEN x)` divide x by its "leading term" so that the series is either 0 or equal to $t^v(1 + O(t))$. Shallow function if the "leading term" is 1.

`int ser_isexactzero(GEN x)` return 1 if x is a zero series, all of whose known coefficients are exact zeroes; this implies that `sign`$(x) = 0$ and `lg`$(x) \leq 3$.

`GEN ser_inv(GEN x)` return the inverse of the `t_SER` x using Newton iteration.

`GEN psi1series(long n, long v, long prec)` creates the `t_SER` $\psi(1 + x + O(x^n))$ in variable v.

10.8 Functions to handle `t_FFELT`.

These functions define the public interface of the `t_FFELT` type to use in generic functions. However, in specific functions, it is better to use the functions class `FpXQ` and/or `Flxq` as appropriate.

`GEN FF_p(GEN a)` returns the characteristic of the definition field of the `t_FFELT` element `a`.

`long FF_f(GEN a)` returns the dimension of the definition field over its prime field; the cardinality of the dimension field is thus p^f.

`GEN FF_p_i(GEN a)` shallow version of `FF_p`.

`GEN FF_q(GEN a)` returns the cardinality of the definition field of the `t_FFELT` element `a`.

`GEN FF_mod(GEN a)` returns the polynomial (with reduced `t_INT` coefficients) defining the finite field, in the variable used to display a.

`long FF_var(GEN a)` returns the variable used to display a.

`GEN FF_gen(GEN a)` returns the standard generator of the definition field of the `t_FFELT` element `a`, see `ffgen`, that is $x \pmod T$ where T is the polynomial over the prime field that define the finite field.

`GEN FF_to_FpXQ(GEN a)` converts the `t_FFELT` `a` to a polynomial P with reduced `t_INT` coefficients such that $a = P(g)$ where g is the generator of the finite field returned by `ffgen`, in the variable used to display g.

`GEN FF_to_FpXQ_i(GEN a)` shallow version of `FF_to_FpXQ`.

`GEN FF_to_F2xq(GEN a)` converts the `t_FFELT` `a` to a `F2x` P such that $a = P(g)$ where g is the generator of the finite field returned by `ffgen`, in the variable used to display g. This only work if the characteristic is 2.

GEN FF_to_F2xq_i(GEN a) shallow version of FF_to_F2xq.

GEN FF_to_Flxq(GEN a) converts the t_FFELT a to a Flx P such that $a = P(g)$ where g is the generator of the finite field returned by ffgen, in the variable used to display g. This only work if the characteristic is small enough.

GEN FF_to_Flxq_i(GEN a) shallow version of FF_to_Flxq.

GEN p_to_FF(GEN p, long v) returns a t_FFELT equal to 1 in the finite field $\mathbf{Z}/p\mathbf{Z}$. Useful for generic code that wants to handle (inefficiently) $\mathbf{Z}/p\mathbf{Z}$ as if it were not a prime field.

GEN Tp_to_FF(GEN T, GEN p) returns a t_FFELT equal to 1 in the finite field $\mathbf{F}_p/(T)$, where T is a ZX, assumed to be irreducible modulo p, or NULL in which case the routine acts as p_to_FF(p,0). No checks.

GEN Fq_to_FF(GEN x, GEN ff) returns a t_FFELT equal to x in the finite field defined by the t_FFELT ff, where x is an Fq (either a t_INT or a ZX: a t_POL with t_INT coefficients). No checks.

GEN FqX_to_FFX(GEN x, GEN ff) given an FqX x, return the polynomial with t_FFELT coefficients obtained by applying Fq_to_FF coefficientwise. No checks, and no normalization if the leading coefficient maps to 0.

GEN FF_1(GEN a) returns the unity in the definition field of the t_FFELT element a.

GEN FF_zero(GEN a) returns the zero element of the definition field of the t_FFELT element a.

int FF_equal0(GEN a) returns 1 if the t_FFELT a is equal to 0 else returns 0.

int FF_equal1(GEN a) returns 1 if the t_FFELT a is equal to 1 else returns 0.

int FF_equalm1(GEN a) returns -1 if the t_FFELT a is equal to 1 else returns 0.

int FF_equal(GEN a, GEN b) return 1 if the t_FFELT a and b have the same definition field and are equal, else 0.

int FF_samefield(GEN a, GEN b) return 1 if the t_FFELT a and b have the same definition field, else 0.

int Rg_is_FF(GEN c, GEN *ff) to be called successively on many objects, setting *ff = NULL (unset) initially. Returns 1 as long as c is a t_FFELT defined over the same field as *ff (setting *ff = c if unset), and 0 otherwise.

int RgC_is_FFC(GEN x, GEN *ff) apply Rg_is_FF successively to all components of the t_VEC or t_COL x. Return 0 if one call fails, and 1 otherwise.

int RgM_is_FFM(GEN x, GEN *ff) apply Rg_is_FF to all components of the t_MAT. Return 0 if one call fails, and 1 otherwise.

GEN FF_add(GEN a, GEN b) returns $a + b$ where a and b are t_FFELT having the same definition field.

GEN FF_Z_add(GEN a, GEN x) returns $a + x$, where a is a t_FFELT, and x is a t_INT, the computation being performed in the definition field of a.

GEN FF_Q_add(GEN a, GEN x) returns $a + x$, where a is a t_FFELT, and x is a t_RFRAC, the computation being performed in the definition field of a.

GEN FF_sub(GEN a, GEN b) returns $a - b$ where a and b are t_FFELT having the same definition field.

GEN FF_mul(GEN a, GEN b) returns ab where a and b are t_FFELT having the same definition field.

GEN FF_Z_mul(GEN a, GEN b) returns ab, where a is a t_FFELT, and b is a t_INT, the computation being performed in the definition field of a.

GEN FF_div(GEN a, GEN b) returns a/b where a and b are t_FFELT having the same definition field.

GEN FF_neg(GEN a) returns $-a$ where a is a t_FFELT.

GEN FF_neg_i(GEN a) shallow function returning $-a$ where a is a t_FFELT.

GEN FF_inv(GEN a) returns a^{-1} where a is a t_FFELT.

GEN FF_sqr(GEN a) returns a^2 where a is a t_FFELT.

GEN FF_mul2n(GEN a, long n) returns $a2^n$ where a is a t_FFELT.

GEN FF_pow(GEN a, GEN n) returns a^n where a is a t_FFELT and n is a t_INT.

GEN FF_Frobenius(GEN a, GEN n) returns x^{p^n} where x is the standard generator of the definition field of the t_FFELT element a, t_FFELT, n is a t_INT, and p is the characteristic of the definition field of a.

GEN FF_Z_Z_muldiv(GEN a, GEN x, GEN y) returns ay/z, where a is a t_FFELT, and x and y are t_INT, the computation being performed in the definition field of a.

GEN Z_FF_div(GEN x, GEN a) return x/a where a is a t_FFELT, and x is a t_INT, the computation being performed in the definition field of a.

GEN FF_norm(GEN a) returns the norm of the t_FFELT a with respect to its definition field.

GEN FF_trace(GEN a) returns the trace of the t_FFELT a with respect to its definition field.

GEN FF_conjvec(GEN a) returns the vector of conjugates $[a, a^p, a^{p^2}, \ldots, a^{p^{n-1}}]$ where the t_FFELT a belong to a field with p^n elements.

GEN FF_charpoly(GEN a) returns the characteristic polynomial) of the t_FFELT a with respect to its definition field.

GEN FF_minpoly(GEN a) returns the minimal polynomial of the t_FFELT a.

GEN FF_sqrt(GEN a) returns an t_FFELT b such that $a = b^2$ if it exist, where a is a t_FFELT.

long FF_issquareall(GEN x, GEN *pt) returns 1 if x is a square, and 0 otherwise. If x is indeed a square, set pt to its square root.

long FF_issquare(GEN x) returns 1 if x is a square and 0 otherwise.

long FF_ispower(GEN x, GEN K, GEN *pt) Given K a positive integer, returns 1 if x is a K-th power, and 0 otherwise. If x is indeed a K-th power, set pt to its K-th root.

GEN FF_sqrtn(GEN a, GEN n, GEN *zn) returns an n-th root of a if it exist. If zn is non-NULL set it to a primitive n-th root of the unity.

GEN FF_log(GEN a, GEN g, GEN o) the t_FFELT g being a generator for the definition field of the t_FFELT a, returns a t_INT e such that $a^e = g$. If e does not exists, the result is currently undefined. If o is not NULL it is assumed to be a factorization of the multiplicative order of g (as set by FF_primroot)

GEN FF_order(GEN a, GEN o) returns the order of the t_FFELT a. If o is non-NULL, it is assumed that o is a multiple of the order of a.

GEN FF_primroot(GEN a, GEN *o) returns a generator of the multiplicative group of the definition field of the t_FFELT a. If o is not NULL, set it to the factorization of the order of the primitive root (to speed up FF_log).

GEN FF_map(GEN m, GEN a) returns $A(m)$ where A=a.pol assuming a and m belongs to fields having the same characteristic.

10.8.1 FFX.

The functions in this sections take polynomial arguments and a t_FFELT a. The coefficients of the polynomials must be of type t_INT, t_INTMOD or t_FFELT and compatible with a.

GEN FFX_add(GEN P, GEN Q, GEN a) returns the sum of the polynomials P and Q defined over the definition field of the t_FFELT a.

GEN FFX_mul(GEN P, GEN Q, GEN a) returns the product of the polynomials P and Q defined over the definition field of the t_FFELT a.

GEN FFX_sqr(GEN P, GEN a) returns the square of the polynomial P defined over the definition field of the t_FFELT a.

GEN FFX_rem(GEN P, GEN Q, GEN a) returns the remainder of the polynomial P modulo the polynomial Q, where P and Q are defined over the definition field of the t_FFELT a.

GEN FFX_gcd(GEN P, GEN Q, GEN a) returns the GCD of the polynomials P and Q defined over the definition field of the t_FFELT a.

GEN FFX_extgcd(GEN P, GEN Q, GEN a, GEN *U, GEN *V) returns the GCD of the polynomials P and Q defined over the definition field of the t_FFELT a and sets *U, *V to the Bezout coefficients such that $*\mathtt{U}P + *\mathtt{V}Q = d$. If *U is set to NULL, it is not computed which is a bit faster.

GEN FFX_halfgcd(GEN x, GEN y, GEN a) returns a two-by-two matrix M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

GEN FFX_resultant(GEN P, GEN Q, GEN a) returns the resultant of the polynomials P and Q where P and Q are defined over the definition field of the t_FFELT a.

GEN FFX_disc(GEN P, GEN a) returns the discriminant of the polynomial P where P is defined over the definition field of the t_FFELT a.

GEN FFX_ispower(GEN P, ulong k, GEN a, GEN *py) return 1 if the FFX P is a k-th power, 0 otherwise, where P is defined over the definition field of the t_FFELT a. If py is not NULL, set it to g such that $g^k = f$.

GEN FFX_factor(GEN f, GEN a) returns the factorization of the univariate polynomial f over the definition field of the t_FFELT a. The coefficients of f must be of type t_INT, t_INTMOD or t_FFELT and compatible with a.

GEN FFX_factor_squarefree(GEN f, GEN a) returns the squarefree factorization of the univariate polynomial f over the definition field of the t_FFELT a. This is a vector $[u_1, \ldots, u_k]$ of pairwise coprime FFX such that $u_k \neq 1$ and $f = \prod u_i^i$.

GEN FFX_ddf(GEN f, GEN a) assuming that f is squarefree, returns the distinct degree factorization of f modulo p. The returned value v is a t_VEC with two components: F=v[1] is a vector of (FFX) factors, and E=v[2] is a t_VECSMALL, such that f is equal to the product of the F[i] and each F[i] is a product of irreducible factors of degree E[i].

GEN FFX_degfact(GEN f, GEN a), as FFX_factor, but the degrees of the irreducible factors are returned instead of the factors themselves (as a t_VECSMALL).

GEN FFX_roots(GEN f, GEN a) returns the roots (t_FFELT) of the univariate polynomial f over the definition field of the t_FFELT a. The coefficients of f must be of type t_INT, t_INTMOD or t_FFELT and compatible with a.

GEN FFX_preimagerel(GEN F, GEN x, GEN a) returns $P\%F$ where P=x.pol assuming a and x belongs to fields having the same characteristic, and that the coefficients of F belong to the definition field of a.

GEN FFX_preimage(GEN F, GEN x, GEN a) as FFX_preimagerel but return NULL if the remainder is of degree greater or equal to 1, the constant coefficient otherwise.

10.8.2 FFM.

GEN FFM_FFC_gauss(GEN M, GEN C, GEN ff) given a matrix M (t_MAT) and a column vector C (t_COL) over the finite field given by ff (t_FFELT) such that M is invertible, return the unique column vector X such that $MX = C$.

GEN FFM_FFC_invimage(GEN M, GEN C, GEN ff) given a matrix M (t_MAT) and a column vector C (t_COL) over the finite field given by ff (t_FFELT), return a column vector X such that $MX = C$, or NULL if no such vector exists.

GEN FFM_FFC_mul(GEN M, GEN C, GEN ff) returns the product of the matrix M (t_MAT) and the column vector C (t_COL) over the finite field given by ff (t_FFELT).

GEN FFM_deplin(GEN M, GEN ff) returns a nonzero vector (t_COL) in the kernel of the matrix M over the finite field given by ff, or NULL if no such vector exists.

GEN FFM_det(GEN M, GEN ff) returns the determinant of the matrix M over the finite field given by ff.

GEN FFM_gauss(GEN M, GEN N, GEN ff) given two matrices M and N (t_MAT) over the finite field given by ff (t_FFELT) such that M is invertible, return the unique matrix X such that $MX = N$.

GEN FFM_image(GEN M, GEN ff) returns a matrix whose columns span the image of the matrix M over the finite field given by ff.

GEN FFM_indexrank(GEN M, GEN ff) given a matrix M of rank r over the finite field given by ff, returns a vector with two t_VECSMALL components y and z containing r row and column indices, respectively, such that the $r \times r$-matrix formed by the M[i,j] for i in y and j in z is invertible.

GEN FFM_inv(GEN M, GEN ff) returns the inverse of the square matrix M over the finite field given by ff, or NULL if M is not invertible.

GEN FFM_invimage(GEN M, GEN N, GEN ff) given two matrices M and N (t_MAT) over the finite field given by ff (t_FFELT), return a matrix X such that $MX = N$, or NULL if no such matrix exists.

GEN FFM_ker(GEN M, GEN ff) returns the kernel of the t_MAT M over the finite field given by the t_FFELT ff.

GEN FFM_mul(GEN M, GEN N, GEN ff) returns the product of the matrices M and N (t_MAT) over the finite field given by ff (t_FFELT).

long FFM_rank(GEN M, GEN ff) returns the rank of the matrix M over the finite field given by ff.

GEN FFM_suppl(GEN M, GEN ff) given a matrix M over the finite field given by ff whose columns are linearly independent, returns a square invertible matrix whose first columns are those of M.

10.8.3 FFXQ.

GEN FFXQ_mul(GEN P, GEN Q, GEN T, GEN a) returns the product of the polynomials P and Q modulo the polynomial T, where P, Q and T are defined over the definition field of the t_FFELT a.

GEN FFXQ_sqr(GEN P, GEN T, GEN a) returns the square of the polynomial P modulo the polynomial T, where P and T are defined over the definition field of the t_FFELT a.

GEN FFXQ_inv(GEN P, GEN Q, GEN a) returns the inverse of the polynomial P modulo the polynomial Q, where P and Q are defined over the definition field of the t_FFELT a.

GEN FFXQ_minpoly(GEN Pf, GEN Qf, GEN a) returns the minimal polynomial of the polynomial P modulo the polynomial Q, where P and Q are defined over the definition field of the t_FFELT a.

10.9 Transcendental functions.

The following two functions are only useful when interacting with gp, to manipulate its internal default precision (expressed as a number of decimal digits, not in words as used everywhere else):

long getrealprecision(void) returns realprecision.

long setrealprecision(long n, long *prec) sets the new realprecision to n, which is returned. As a side effect, set prec to the corresponding number of words ndec2prec(n).

10.9.1 Transcendental functions with t_REAL arguments.

In the following routines, x is assumed to be a t_REAL and the result is a t_REAL (sometimes a t_COMPLEX with t_REAL components), with the largest accuracy which can be deduced from the input. The naming scheme is inconsistent here, since we sometimes use the prefix mp even though t_INT inputs are forbidden:

GEN sqrtr(GEN x) returns the square root of x.

GEN cbrtr(GEN x) returns the real cube root of x.

GEN sqrtnr(GEN x, long n) returns the n-th root of x, assuming $n \geq 1$ and $x \geq 0$.

GEN sqrtnr_abs(GEN x, long n) returns the n-th root of $|x|$, assuming $n \geq 1$ and $x \neq 0$.

GEN mpcos[z](GEN x[, GEN z]) returns $\cos(x)$.

GEN mpsin[z](GEN x[, GEN z]) returns $\sin(x)$.

GEN mplog[z](GEN x[, GEN z]) returns $\log(x)$. We must have $x > 0$ since the result must be a t_REAL. Use glog for the general case, where you want such computations as $\log(-1) = I$.

GEN mpexp[z](GEN x[, GEN z]) returns $\exp(x)$.

GEN mpexpm1(GEN x) returns $\exp(x) - 1$, but is more accurate than subrs(mpexp(x), 1), which suffers from catastrophic cancellation if $|x|$ is very small.

void mpsincosm1(GEN x, GEN *s, GEN *c) sets s and c to $\sin(x)$ and $\cos(x)-1$ respectively, where x is a t_REAL; the latter is more accurate than subrs(mpcos(y), 1), which suffers from catastrophic cancellation if $|x|$ is very small.

GEN mpveceint1(GEN C, GEN eC, long n) as veceint1; assumes that $C > 0$ is a t_REAL and that eC is NULL or mpexp(C).

GEN mpeint1(GEN x, GEN expx) returns eint1(x), for a t_REAL $x \neq 0$, assuming that expx is mpexp(x).

A few variants on the Lambert function: they actually work when gtofp can map all GEN arguments to a t_REAL.

GEN mplambertW(GEN y) solution $x = W_0(y)$ of the implicit equation $x\exp(x) = y$, for $y > -1/e$ a t_REAL.

GEN mplambertx_logx(GEN a, GEN b, long bit) solve $x - a\log(x) = b$ with $a > 0$ and $b \geq a(1-\log(a))$.

GEN mplambertX(GEN y, long bit) as mplambertx_logx in the special case $a = 1$, $b = \log(y)$. In other words, solve $e^x/x = y$ with $y \geq e$.

GEN mplambertxlogx_x(GEN a, GEN b, long bit) solve $x\log(x) - ax = b$; if $b < 0$, assume $a \geq 1 + \log|b|$.

Useful low-level functions which *disregard* the sign of x:

GEN sqrtr_abs(GEN x) returns $\sqrt{|x|}$ assuming $x \neq 0$.

GEN cbrtr_abs(GEN x) returns $|x|^{1/3}$ assuming $x \neq 0$.

GEN exp1r_abs(GEN x) returns $\exp(|x|) - 1$, assuming $x \neq 0$.

GEN logr_abs(GEN x) returns $\log(|x|)$, assuming $x \neq 0$.

10.9.2 Other complex transcendental functions.

GEN atanhuu(ulong u, ulong v, long prec) computes $\operatorname{atanh}(u/v)$ using binary splitting, assuming $0 < u < v$. Not memory clean but suitable for gerepileupto.

GEN atanhui(ulong u, GEN v, long prec) computes $\operatorname{atanh}(u/v)$ using binary splitting, assuming $0 < u < v$. Not memory clean but suitable for gerepileupto.

GEN szeta(long s, long prec) returns the value of Riemann's zeta function at the (possibly negative) integer $s \neq 1$, in relative accuracy prec.

GEN veczeta(GEN a, GEN b, long N, long prec) returns in a vector all the $\zeta(aj + b)$, where $j = 0, 1, \ldots, N-1$, where a and b are real numbers (of arbitrary type, although t_INT is treated more efficiently) and $b > 1$. Assumes that $N \geq 1$.

GEN ggamma1m1(GEN x, long prec) return $\Gamma(1 + x) - 1$ assuming $|x| < 1$. Guard against cancellation when x is small.

A few variants on sin and cos:

void mpsincos(GEN x, GEN *s, GEN *c) sets s and c to $\sin(x)$ and $\cos(x)$ respectively, where x is a t_REAL

void mpsinhcosh(GEN x, GEN *s, GEN *c) sets s and c to $\sinh(x)$ and $\cosh(x)$ respectively, where x is a t_REAL

GEN expIr(GEN x) returns $\exp(ix)$, where x is a t_REAL. The return type is t_COMPLEX unless the imaginary part is equal to 0 to the current accuracy (its sign is 0).

GEN expIPiR(GEN x, long prec) return $\exp(i\pi x)$, where x is a real number (t_INT, t_FRAC or t_REAL).

GEN expIPiC(GEN z, long prec) return $\exp(i\pi x)$, where x is a complex number (t_INT, t_FRAC, t_REAL or t_COMPLEX).

GEN expIxy(GEN x, GEN y, long prec) returns $\exp(ixy)$. Efficient when x is real and y pure imaginary.

GEN pow2Pis(GEN s, long prec) returns $(2\pi)^s$. The intent of this function and the next ones is to be accurate even if s has a huge imaginary part: π is computed at an accuracy taking into account the cancellation induced by argument reduction when computing the sine or cosine of $\Im s \log 2\pi$.

GEN powPis(GEN s, long prec) returns π^s, as pow2Pis.

long powcx_prec(long e, GEN s, long prec) if $e \approx \log_2 |x|$ return the precision at which $\log(x)$ must be computed to evaluate x^s reliably (taking into account argument reduction).

GEN powcx(GEN x, GEN logx, GEN s, long prec) assuming s is a t_COMPLEX and logx is $\log(x)$ computed to accuracy powcx_prec, return x^s.

void gsincos(GEN x, GEN *s, GEN *c, long prec) general case.

GEN rootsof1_cx(GEN d, long prec) return $e(1/d)$ at precision prec, $e(x) = \exp(2i\pi x)$.

GEN rootsof1u_cx(ulong d, long prec) return $e(1/d)$ at precision prec.

GEN rootsof1q_cx(long a, long b, long prec) return $e(a/b)$ at precision prec.

GEN rootsof1powinit(long a, long b, long prec) precompute b-th roots of 1 for rootsof1pow, i.e. to later compute $e(ac/b)$ for varying c.

GEN rootsof1pow(GEN T, long c) given $T =$ rootsof1powinit($a, b,$ prec), return $e(ac/b)$.

A generalization of affrr_fixlg

GEN affc_fixlg(GEN x, GEN res) assume res was allocated using cgetc, and that x is either a t_REAL or a t_COMPLEX with t_REAL components. Assign x to res, first shortening the components of res if needed (in a gerepile-safe way). Further convert res to a t_REAL if x is a t_REAL.

GEN trans_eval(const char *fun, GEN (*f) (GEN, long), GEN x, long prec) evaluate the transcendental function f (named "fun" at the argument x and precision prec. This is a quick way to implement a transcendental function to be made available under GP, starting from a C function handling only t_REAL and t_COMPLEX arguments. This routine first converts x to a suitable type:

- t_INT/t_FRAC to t_REAL of precision prec, t_QUAD to t_REAL or t_COMPLEX of precision prec.

- t_POLMOD to a t_COL of complex embeddings (as in conjvec)

Then evaluates the function at t_VEC, t_COL, t_MAT arguments coefficientwise.

GEN trans_evalgen(const char *fun, void *E, GEN (*f)(void*, GEN, long), GEN x, long prec), general variant evaluating $f(E, x, prec)$, where the function prototype allows to wrap an arbitrary context given by the argument E.

10.9.3 Modular functions.

`GEN cxredsl2(GEN z, GEN *g)` given t a `t_COMPLEX` belonging to the upper half-plane, find $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\gamma \cdot z$ belongs to the standard fundamental domain and set `*g` to γ.

`GEN cxredsl2_i(GEN z, GEN *g, GEN *czd)` as `cxredsl2`; also sets `*czd` to $cz+d$, if $\gamma = [a, b; c, d]$.

`GEN cxEk(GEN tau, long k, long prec)` returns $E_k(\tau)$ by direct evaluation of $1 + 2/\zeta(1-k)\sum_n n^{k-1}q^n/(1-q^n)$, $q = e(\tau)$. Assume that $\Im\tau > 0$ and k even. Very slow unless τ is already reduced modulo $\mathrm{SL}_2(\mathbf{Z})$. Not `gerepile`-clean but suitable for `gerepileupto`.

10.9.4 Transcendental functions with `t_PADIC` arguments.

The argument x is assumed to be a `t_PADIC`.

`GEN Qp_exp(GEN x)` shortcut for `gexp(x, /*ignored*/prec)`

`long Qp_exp_prec(GEN x)` number of terms to sum in the $\exp(x)$ series to reach the same p-adic accuracy as $x \neq 0$. If $n = p - 1$, $e = v_p(x)$ and $b =$ `precp`(x), this is the ceiling of $nb/(ne - 1)$. Return -1 if the series does not converge ($ne \leq 1$).

`GEN Qp_gamma(GEN x)` shortcut for `ggamma(x, /*ignored*/prec)`

`GEN Qp_zeta(GEN x)` shortcut for `gzeta(x, /*ignored*/prec)`; assume that $x \neq 1$.

`GEN Qp_log(GEN x)` shortcut for `glog(x, /*ignored*/prec)`

`GEN Qp_sqrt(GEN x)` shortcut for `gsqrt(x, /*ignored*/prec)` Return `NULL` if x is not a square.

`GEN Qp_sqrtn(GEN x, GEN n, GEN *z)` shortcut for `gsqrtn(x, n, z, /*ignored*/prec)`. Return `NULL` if x is not an n-th power.

`GEN Qp_agm2_sequence(GEN a1, GEN b1)` assume $a_1/b_1 = 1 \bmod p$ if p odd and mod 2^4 if $p = 2$. Let $A_1 = a_1/p^v$ and $B_1 = b_1/p^v$ with $v = v_p(a_1) = v_p(b_1)$; let further $A_{n+1} = (A_n + B_n + 2B_{n+1})/4$, $B_{n+1} = B_n\sqrt{A_n/B_n}$ (the square root of A_nB_n congruent to B_n mod p) and $R_n = p^v(A_n - B_n)$. We stop when R_n is 0 at the given p-adic accuracy. This function returns in a triplet `t_VEC` the three sequences (A_n), (B_n) and (R_n), corresponding to a sequence of 2-isogenies on the Tate curve $y^2 = x(x - a_1)(x + a_1 - b_1)$. The common limit of A_n and B_n is the $M_2(a_1, b_1)$, the square of the p-adic AGM of $\sqrt(a_1)$ and $\sqrt(b_1)$. This is given by `ellQp_Ei` and is used by corresponding ascending and descending p-adic Landen transforms:

`void Qp_ascending_Landen(GEN ABR, GEN *ptx, GEN *pty)`

`void Qp_descending_Landen(GEN ABR, GEN *ptx, GEN *pty)`

10.9.5 Cached constants.

The cached constant is returned at its current precision, which may be larger than `prec`. One should always use the `mp`*xxx* variant: `mppi`, `mpeuler`, or `mplog2`.

`GEN consteuler(long prec)` precomputes Euler-Mascheroni's constant at precision `prec`.

`GEN constcatalan(long prec)` precomputes Catalan's constant at precision `prec`.

`GEN constpi(long prec)` precomputes π at precision `prec`.

`GEN constlog2(long prec)` precomputes $\log(2)$ at precision `prec`.

`void constbern(long n)` precomputes the n even Bernoulli numbers $B_2, \ldots, B_{2n}$ as `t_FRAC`. No more than n Bernoulli numbers will ever be stored (by `bernfrac` or `bernreal`), unless a subsequent call to `constbern` increases the cache.

`GEN constzeta(long n, long prec)` ensures that the n values $\gamma, \zeta(2), \ldots, \zeta(n)$ are cached at accuracy bigger than or equal to `prec` and return a vector containing at least those value. Note that $\gamma = \lim_1 \zeta(s) - 1/(s-1)$. If the accuracy of cached data is too low or n is greater than the cache length, the cache is recomputed at the given parameters.

The following functions use cached data if `prec` is smaller than the precision of the cached value; otherwise the newly computed data replaces the old cache.

`GEN mppi(long prec)` returns π at precision `prec`.

`GEN Pi2n(long n, long prec)` returns $2^n\pi$ at precision `prec`.

`GEN PiI2(long n, long prec)` returns the complex number $2\pi i$ at precision `prec`.

`GEN PiI2n(long n, long prec)` returns the complex number $2^n\pi i$ at precision `prec`.

`GEN mpeuler(long prec)` returns Euler-Mascheroni's constant at precision `prec`.

`GEN mpeuler(long prec)` returns Catalan's number at precision `prec`.

`GEN mplog2(long prec)` returns $\log 2$ at precision `prec`.

The following functions use the Bernoulli numbers cache initialized by `constbern`:

`GEN bernreal(long i, long prec)` returns the Bernoulli number B_i as a `t_REAL` at precision `prec`. If `constbern(n)` was called previously with $n \geq i$, then the cached value is (converted to a `t_REAL` of accuracy `prec` then) returned. Otherwise, the missing value is computed; the cache is not updated.

`GEN bernfrac(long i)` returns the Bernoulli number B_i as a rational number (`t_FRAC` or `t_INT`). If the `constbern` cache includes B_i, the latter is returned. Otherwise, the missing value is computed; the cache is not updated.

10.9.6 Obsolete functions.

`void mpbern(long n, long prec)`

10.10 Permutations .

Permutations are represented in two different ways

- (`perm`) a `t_VECSMALL` p representing the bijection $i \mapsto p[i]$; unless mentioned otherwise, this is the form used in the functions below for both input and output,
- (`cyc`) a `t_VEC` of `t_VECSMALL`s representing a product of disjoint cycles.

`GEN identity_perm(long n)` return the identity permutation on n symbols.

`GEN cyclic_perm(long n, long d)` return the cyclic permutation mapping i to $i + d \pmod n$ in S_n. Assume that $d \leq n$.

`GEN perm_mul(GEN s, GEN t)` multiply s and t (composition $s \circ t$)

`GEN perm_sqr(GEN s)` multiply s by itself (composition $s \circ s$)

`GEN perm_conj(GEN s, GEN t)` return sts^{-1}.

`int perm_commute(GEN p, GEN q)` return 1 if p and q commute, 0 otherwise.

`GEN perm_inv(GEN p)` returns the inverse of p.

`GEN perm_pow(GEN p, GEN n)` returns p^n

`GEN perm_powu(GEN p, ulong n)` returns p^n

`GEN cyc_pow_perm(GEN p, long n)` the permutation p is given as a product of disjoint cycles (`cyc`); return p^n (as a `perm`).

`GEN cyc_pow(GEN p, long n)` the permutation p is given as a product of disjoint cycles (`cyc`); return p^n (as a `cyc`).

`GEN perm_cycles(GEN p)` return the cyclic decomposition of p.

`GEN perm_order(GEN p)` returns the order of the permutation p (as the lcm of its cycle lengths).

`ulong perm_orderu(GEN p)` returns the order of the permutation p (as the lcm of its cycle lengths) assuming it fits in a `ulong`.

`long perm_sign(GEN p)` returns the sign of the permutation p.

`GEN vecperm_orbits(GEN gen, long n)` return the orbits of $\{1, 2, \ldots, n\}$ under the action of the subgroup of S_n generated by *gen*.

`GEN Z_to_perm(long n, GEN x)` as `numtoperm`, returning a `t_VECSMALL`.

`GEN perm_to_Z(GEN v)` as `permtonum` for a `t_VECSMALL` input.

`GEN perm_to_GAP(GEN p)` return a `t_STR` which is a representation of p comptatible with the GAP computer algebra system.

10.11 Small groups.

The small (finite) groups facility is meant to deal with subgroups of Galois groups obtained by `galoisinit` and thus is currently limited to weakly super-solvable groups.

A group *grp* of order n is represented by its regular representation (for an arbitrary ordering of its element) in S_n. A subgroup of such group is represented by the restriction of the representation to the subgroup. A *small group* can be either a group or a subgroup. Thus it is embedded in some S_n, where n is the multiple of the order. Such an n is called the *domain* of the small group. The domain of a trivial subgroup cannot be derived from the subgroup data, so some functions require the subgroup domain as argument.

The small group *grp* is represented by a `t_VEC` with two components:

grp[1] is a generating subset $[s_1, \ldots, s_g]$ of *grp* expressed as a vector of permutations of length n.

grp[2] contains the relative orders $[o_1, \ldots, o_g]$ of the generators *grp*[1].

See `galoisinit` for the technical details.

`GEN checkgroup(GEN gal, GEN *elts)` check whether *gal* is a small group or a Galois group. Returns the underlying small group and set *elts* to the list of elements or to `NULL` if it is not known.

`GEN checkgroupelts(GEN gal)` check whether *gal* is a small group or a Galois group, or a vector of permutations listing the group elements. Returns the list of group elements as permutations.

`GEN galois_group(GEN gal)` return the underlying small group of the Galois group *gal*.

`GEN cyclicgroup(GEN g, long s)` return the cyclic group with generator g of order s.

`GEN trivialgroup(void)` return the trivial group.

`GEN dicyclicgroup(GEN g1, GEN g2, long s1, long s2)` returns the group with generators *g1*, *g2* with respecting relative orders *s1*, *s2*.

`GEN abelian_group(GEN v)` let v be a `t_VECSMALL` seen as the SNF of a small abelian group, return its regular representation.

`long group_domain(GEN grp)` returns the `domain` of the *nontrivial* small group *grp*. Return an error if *grp* is trivial.

`GEN group_elts(GEN grp, long n)` returns the list of elements of the small group *grp* of domain n as permutations.

`GEN groupelts_to_group(GEN elts)`, where *elts* is the list of elements of a group, returns the corresponding small group, if it exists, otherwise return `NULL`.

`GEN group_set(GEN grp, long n)` returns a *F2v* b such that $b[i]$ is set if and only if the small group *grp* of domain n contains a permutation sending 1 to i.

`GEN groupelts_set(GEN elts, long n)`, where *elts* is the list of elements of a small group of domain n, returns a *F2v* b such that $b[i]$ is set if and only if the small group contains a permutation sending 1 to i.

`GEN groupelts_conj_set(GEN elts, GEN p)`, where *elts* is the list of elements of a small group of domain n, returns a *F2v* b such that $b[i]$ is set if and only if the small group contains a permutation sending $p^{-1}[1]$ to $p^{-1}[i]$.

`int group_subgroup_is_faithful(GEN G, GEN H)` return 1 if the action of G on G/H by translation is faithful, 0 otherwise.

`GEN groupelts_conjclasses(GEN elts, long *pn)`, where *elts* is the list of elements of a small group (sorted with respect to `vecsmall_lexcmp`), return a `t_VECSMALL` `conj` of the same length such that `conj[i]` is the index in $\{1,\cdots,n\}$ of the conjugacy class of `elts[i]` for some unspecified but deterministic ordering of the classes, where n is the number of conjugacy classes. If `pn` is non `NULL`, `*pn` is set to n.

`GEN conjclasses_repr(GEN conj, long nb)`, where `conj` and `nb` are as returned by the call `groupelts_conjclasses(elts)`, return `t_VECSMALL` of length `nb` which gives the indices in `elts` of a representative of each conjugacy class.

`GEN group_to_cc(GEN G)`, where G is a small group or a Galois group, returns a `cc` (conjclasses) structure `[elts,conj,rep,flag]`, as obtained by `alggroupcenter`, where `conj` is `groupelts_conjclasses(elts)` and `rep` is the attached `conjclasses_repr`. `flag` is 1 if the permutation representation is transitive (in which case an element g of G is characterized by $g[1]$), and 0 otherwise. Shallow function.

`long group_order(GEN grp)` returns the order of the small group *grp* (which is the product of the relative orders).

`long group_isabelian(GEN grp)` returns 1 if the small group *grp* is Abelian, else 0.

`GEN group_abelianHNF(GEN grp, GEN elts)` if *grp* is not Abelian, returns `NULL`, else returns the HNF matrix of *grp* with respect to the generating family *grp*[1]. If *elts* is no `NULL`, it must be the list of elements of *grp*.

`GEN group_abelianSNF(GEN grp, GEN elts)` if *grp* is not Abelian, returns `NULL`, else returns its cyclic decomposition. If *elts* is no `NULL`, it must be the list of elements of *grp*.

`long group_subgroup_isnormal(GEN G, GEN H)`, H being a subgroup of the small group G, returns 1 if H is normal in G, else 0.

`long group_isA4S4(GEN grp)` returns 1 if the small group *grp* is isomorphic to A_4, 2 if it is isomorphic to S_4, 3 if it is isomorphic to $(3 \times 3) : 4$ and 0 else. This is mainly to deal with the idiosyncrasy of the format.

`GEN group_leftcoset(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, the left coset gG as a vector of permutations.

`GEN group_rightcoset(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, the right coset Gg as a vector of permutations.

`long group_perm_normalize(GEN G, GEN g)` where G is a small group and g a permutation of the same domain, return 1 if $gGg^{-1} = G$, else 0.

`GEN group_quotient(GEN G, GEN H)`, where G is a small group and H is a subgroup of G, returns the quotient map $G \to G/H$ as an abstract data structure.

`GEN groupelts_quotient(GEN elts, GEN H)`, where *elts* is the list of elements of a small group G, H is a subgroup of G, returns the quotient map $G \to G/H$ as an abstract data structure.

`GEN quotient_perm(GEN C, GEN g)` where C is the quotient map $G \to G/H$ for some subgroup H of G and g an element of G, return the image of g by C (i.e. the coset gH).

`GEN quotient_group(GEN C, GEN G)` where C is the quotient map $G \to G/H$ for some *normal* subgroup H of G, return the quotient group G/H as a small group.

`GEN quotient_groupelts(GEN C)` where C is the quotient map $G \to G/H$ for some group G and some *normal* subgroup H of G, return the list of elements of the quotient group G/H (as permutations over corresponding to the regular representation).

`GEN quotient_subgroup_lift(GEN C, GEN H, GEN S)` where C is the quotient map $G \to G/H$ for some group G normalizing H and S is a subgroup of G/H, return the inverse image of S by C.

`GEN group_subgroups(GEN grp)` returns the list of subgroups of the small group *grp* as a `t_VEC`.

`GEN groupelts_solvablesubgroups(GEN elts)` where *elts* is the list of elements of a finite group, returns the list of its solvable subgroups, each as a list of its elements.

`GEN subgroups_tableset(GEN S, long n)` where S is a vector of subgroups of domain n, returns a table which matchs the set of elements of the subgroups against the index of the subgroups.

`long tableset_find_index(GEN tbl, GEN set)` searchs the set `set` in the table `tbl` and returns its attached index, or 0 if not found.

`GEN groupelts_abelian_group(GEN elts)` where *elts* is the list of elements of an *Abelian* small group, returns the corresponding small group.

`long groupelts_exponent(GEN elts)` where *elts* is the list of elements of a small group, returns the exponent the group (the LCM of the order of the elements of the group).

`GEN groupelts_center(GEN elts)` where *elts* is the list of elements of a small group, returns the list of elements of the center of the group.

`GEN group_export(GEN grp, long format)` convert a small group to another format, as a `t_STR` describing the group for the given syntax, see `galoisexport`.

`GEN group_export_GAP(GEN G)` export a small group to GAP format.

`GEN group_export_MAGMA(GEN G)` export a small group to MAGMA format.

`long group_ident(GEN grp, GEN elts)` returns the index of the small group *grp* in the GAP4 Small Group library, see `galoisidentify`. If *elts* is not `NULL`, it must be the list of elements of *grp*.

`long group_ident_trans(GEN grp, GEN elts)` returns the index of the regular representation of the small group *grp* in the GAP4 Transitive Group library, see `polgalois`. If *elts* is no `NULL`, it must be the list of elements of *grp*.

Chapter 11:
Standard data structures

11.1 Character strings.

11.1.1 Functions returning a char *.

`char* pari_strdup(const char *s)` returns a malloc'ed copy of s (uses `pari_malloc`).

`char* pari_strndup(const char *s, long n)` returns a malloc'ed copy of at most n chars from s (uses `pari_malloc`). If s is longer than n, only n characters are copied and a terminal null byte is added.

`char* stack_strdup(const char *s)` returns a copy of s, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_strcat(const char *s, const char *t)` returns the concatenation of s and t, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_sprintf(const char *fmt, ...)` runs `pari_sprintf` on the given arguments, returning a string allocated on the PARI stack.

`char* uordinal(ulong x)` return the ordinal number attached to x (i.e. 1st, 2nd, etc.) as a `stack_malloc`'ed string.

`char* itostr(GEN x)` writes the `t_INT` x to a `stack_malloc`'ed string.

`char* GENtostr(GEN x)`, using the current default output format (`GP_DATA->fmt`, which contains the output style and the number of significant digits to print), converts x to a malloc'ed string. Simple variant of `pari_sprintf`.

`char* GENtostr_raw(GEN x)` as `GENtostr` with the following differences: 1) the output format is `f_RAW`; 2) the result is allocated on the stack and *must not* be freed.

`char* GENtostr_unquoted(GEN x)` as `GENtostr_raw` with the following additional difference: a `t_STR` x is printed without enclosing quotes (to be used by `print`.

`char* GENtoTeXstr(GEN x)`, as `GENtostr`, except that `f_TEX` overrides the output format from `GP_DATA->fmt`.

`char* RgV_to_str(GEN g, long flag)` g being a vector of `GEN`s, returns a malloc'ed string, the concatenation of the `GENtostr` applied to its elements, except that `t_STR` are printed without enclosing quotes. `flag` determines the output format: `f_RAW`, `f_PRETTYMAT` or `f_TEX`.

11.1.2 Functions returning a t_STR.

GEN strtoGENstr(const char *s) returns a t_STR with content s.

GEN strntoGENstr(const char *s, long n) returns a t_STR containing the first n characters of s.

GEN chartoGENstr(char c) returns a t_STR containing the character c.

GEN GENtoGENstr(GEN x) returns a t_STR containing the printed form of x (in raw format). This is often easier to use that GENtostr (which returns a malloc-ed char*) since there is no need to free the string after use.

GEN GENtoGENstr_nospace(GEN x) as GENtoGENstr, removing all spaces from the output.

GEN Str(GEN g) as RgV_to_str with output format f_RAW, but returns a t_STR, not a malloc'ed string.

GEN strtex(GEN g) as RgV_to_str with output format f_TEX, but returns a t_STR, not a malloc'ed string.

GEN strexpand(GEN g) as RgV_to_str with output format f_RAW, performing tilde and environment expansion on the result. Returns a t_STR, not a malloc'ed string.

GEN gsprintf(const char *fmt, ...) equivalent to pari_sprintf(fmt,..., followed by strtoGENstr. Returns a t_STR, not a malloc'ed string.

GEN gvsprintf(const char *fmt, va_list ap) variadic version of gsprintf

11.1.3 Dynamic strings.

A pari_str is a dynamic string which grows dynamically as needed. This structure contains private data and two public members char *string, which is the string itself and use_stack which tells whether the string lives

• on the PARI stack (value 1), meaning that it will be destroyed by any manipulation of the stack, e.g. a gerepile call or resetting avma;

• in malloc'ed memory (value 0), in which case it is impervious to stack manipulation but will need to be explicitly freed by the user after use, via pari_free(s.string).

void str_init(pari_str *S, int use_stack) initializes a dynamic string; if use_stack is 0, then the string is malloc'ed, else it lives on the PARI stack.

void str_printf(pari_str *S, const char *fmt, ...) write to the end of S the remaining arguments according to PARI format fmt.

void str_putc(pari_str *S, char c) write the character c to the end of S.

void str_puts(pari_str *S, const char *s) write the string s to the end of S.

11.2 Output.

11.2.1 Output contexts.

An output coutext, of type PariOUT, is a struct that models a stream and contains the following function pointers:

```
void (*putch)(char);          /* fputc()-alike */
void (*puts)(const char*);    /* fputs()-alike */
void (*flush)(void);          /* fflush()-alike */
```

The methods putch and puts are used to print a character or a string respectively. The method flush is called to finalize a messages.

The generic functions pari_putc, pari_puts, pari_flush and pari_printf print according to a *default output context*, which should be sufficient for most purposes. Lower level functions are available, which take an explicit output context as first argument:

void out_putc(PariOUT *out, char c) essentially equivalent to out->putc(c). In addition, registers whether the last character printed was a \n.

void out_puts(PariOUT *out, const char *s) essentially equivalent to out->puts(s). In addition, registers whether the last character printed was a \n.

void out_printf(PariOUT *out, const char *fmt, ...)

void out_vprintf(PariOUT *out, const char *fmt, va_list ap)

N.B. The function out_flush does not exist since it would be identical to out->flush()

int pari_last_was_newline(void) returns a nonzero value if the last character printed via out_putc or out_puts was \n, and 0 otherwise.

void pari_set_last_newline(int last) sets the boolean value to be returned by the function pari_last_was_newline to *last*.

11.2.2 Default output context. They are defined by the global variables pariOut and pariErr for normal outputs and warnings/errors, and you probably do not want to change them. If you *do* change them, diverting output in nontrivial ways, this probably means that you are rewriting gp. For completeness, we document in this section what the default output contexts do.

pariOut. writes output to the FILE* pari_outfile, initialized to stdout. The low-level methods are actually the standard putc / fputs, plus some magic to handle a log file if one is open.

pariErr. prints to the FILE* pari_errfile, initialized to stderr. The low-level methods are as above.

You can stick with the default pariOut output context and change PARI's standard output, redirecting pari_outfile to another file, using

void switchout(const char *name) where name is a character string giving the name of the file you want to write to; the output is *appended* at the end of the file. To close the file and revert to outputting to stdout, call switchout(NULL).

11.2.3 PARI colors. In this section we describe the low-level functions used to implement GP's color scheme, attached to the `colors` default. The following symbolic names are attached to gp's output strings:

- `c_ERR` an error message
- `c_HIST` a history number (as in `%1 = ...`)
- `c_PROMPT` a prompt
- `c_INPUT` an input line (minus the prompt part)
- `c_OUTPUT` an output
- `c_HELP` a help message
- `c_TIME` a timer
- `c_NONE` everything else

If the `colors` default is set to a nonempty value, before gp outputs a string, it first outputs an ANSI colors escape sequence — understood by most terminals —, according to the `colors` specifications. As long as this is in effect, the following strings are rendered in color, possibly in bold or underlined.

`void term_color(long c)` prints (as if using `pari_puts`) the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, revert to default printing style.

`void out_term_color(PariOUT *out, long c)` as `term_color`, using output context `out`.

`char* term_get_color(char *s, long c)` returns as a character string the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, the value used to revert to default printing style is returned. The argument `s` is either `NULL` (string allocated on the PARI stack), or preallocated storage (in which case, it must be able to hold at least 16 chars, including the final `\0`).

11.2.4 Obsolete output functions.

These variants of `void output(GEN x)`, which prints `x`, followed by a newline and a buffer flush are complicated to use and less flexible than what we saw above, or than the `pari_printf` variants. They are provided for backward compatibility and are scheduled to disappear.

`void brute(GEN x, char format, long dec)`

`void matbrute(GEN x, char format, long dec)`

`void texe(GEN x, char format, long dec)`

11.3 Files.

The following routines are trivial wrappers around system functions (possibly around one of several functions depending on availability). They are usually integrated within PARI's diagnostics system, printing messages if the debug level for `"files"` is high enough.

`int pari_is_dir(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int pari_is_file(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int file_is_binary(FILE *f)` returns 1 if the file f is a binary file (in the `writebin` sense), 0 otherwise.

`void pari_unlink(const char *s)` deletes the file named s. Warn if the operation fails.

`void pari_fread_chars(void *b, size_t n, FILE *f)` read n chars from stream f, storing the result in pre-allocated buffer b (assumed to be large enough).

`char* path_expand(const char *s)` perform tilde and environment expansion on s. Returns a `malloc`'ed buffer.

`void strftime_expand(const char *s, char *buf, long max)` perform time expansion on s, storing the result (at most `max` chars) in buffer `buf`. Trivial wrapper around

```
time_t t = time(NULL);
strftime(but, max, s, localtime(&t);
```

`char* pari_get_homedir(const char *user)` expands `~user` constructs, returning the home directory of user `user`, or `NULL` if it could not be determined (in particular if the operating system has no such concept). The return value may point to static area and may be overwritten by subsequent system calls: use immediately or `strdup` it.

`int pari_stdin_isatty(void)` returns 1 if our standard input `stdin` is attached to a terminal. Trivial wrapper around `isatty`.

11.3.1 pariFILE.

PARI maintains a linked list of open files, to reclaim resources (file descriptors) on error or interrupts. The corresponding data structure is a `pariFILE`, which is a wrapper around a standard `FILE*`, containing further the file name, its type (regular file, pipe, input or output file, etc.). The following functions create and manipulate this structure; they are integrated within PARI's diagnostics system, printing messages if the debug level for `"files"` is high enough.

`pariFILE* pari_fopen(const char *s, const char *mode)` wrapper around `fopen(s, mode)`, return `NULL` on failure.

`pariFILE* pari_fopen_or_fail(const char *s, const char *mode)` simple wrapper around `fopen(s, mode)`; error on failure.

`pariFILE* pari_fopengz(const char *s)` opens the file whose name is s, and associates a (read-only) `pariFILE` with it. If s is a compressed file (`.gz` suffix), it is uncompressed on the fly. If s cannot be opened, also try to open s`.gz`. Returns `NULL` on failure.

`void pari_fclose(pariFILE *f)` closes the underlying file descriptor and deletes the `pariFILE` struct.

`pariFILE* pari_safefopen(const char *s, const char *mode)` creates a *new* file s (a priori for writing) with `600` permissions. Error if the file already exists. To avoid symlink attacks, a symbolic link exists, regardless of where it points to.

11.3.2 Temporary files.

PARI has its own idea of the system temp directory derived from an environment variable (`$GPTMPDIR`, else `$TMPDIR`), or the first writable directory among `/tmp`, `/var/tmp` and `.`.

`char* pari_unique_dir(const char *s)` creates a "unique directory" and return its name built from the string s, the user id and process pid (on Unix systems). This directory is itself located in the temp directory mentioned above. The name returned is `malloc`'ed.

`char* pari_unique_filename(const char *s)` creates a *new* empty file in the temp directory, whose name contains the id-string s (truncated to its first 8 chars), followed by a system-dependent suffix (incorporating the ids of both the user and the running process, for instance). The function returns the tempfile name and creates an empty file with that name. The name returned is `malloc`'ed.

`char* pari_unique_filename_suffix(const char *s, const char *suf)` analogous to above `pari_unique_filename`, creating a (previously nonexistent) tempfile whose name ends with suffix `suf`.

11.4 Errors.

This section documents the various error classes, and the corresponding arguments to `pari_err`. The general syntax is

`void pari_err(numerr, ...)`

In the sequel, we mostly use sequences of arguments of the form

```
const char *s
const char *fmt, ...
```

where `fmt` is a PARI format, producing a string s from the remaining arguments. Since providing the correct arguments to `pari_err` is quite error-prone, we also provide specialized routines `pari_err_`*ERRORCLASS*`(...)` instead of `pari_err(e_`*ERRORCLASS*`, ...)` so that the C compiler can check their arguments.

We now inspect the list of valid keywords (error classes) for `numerr`, and the corresponding required arguments.

11.4.1 Internal errors, "system" errors.

11.4.1.1 e_ARCH. A requested feature s is not available on this architecture or operating system.

```
pari_err(e_ARCH)
```

prints the error message: `sorry, '`*s*`' not available on this system`.

11.4.1.2 e_BUG. A bug in the PARI library, in function s.

```
pari_err(e_BUG, const char *s)
pari_err_BUG(const char *s)
```

prints the error message: `Bug in `*s*`, please report`.

11.4.1.3 e_FILE. Error while trying to open a file.

```
pari_err(e_FILE, const char *what, const char *name)
pari_err_FILE(const char *what, const char *name)
```

prints the error message: `error opening` *what*`:` `'`*name*`'`.

11.4.1.4 e_FILEDESC. Error while handling a file descriptor.

```
pari_err(e_FILEDESC, const char *where, long n)
pari_err_FILEDESC(const char *where, long n)
```

prints the error message: `invalid file descriptor in` *where*`:` `'`*name*`'`.

11.4.1.5 e_IMPL. A requested feature s is not implemented.

```
pari_err(e_IMPL, const char *s)
pari_err_IMPL(const char *s)
```

prints the error message: `sorry,` s `is not yet implemented`.

11.4.1.6 e_PACKAGE. Missing optional package s.

```
pari_err(e_PACKAGE, const char *s)
pari_err_PACKAGE(const char *s)
```

prints the error message: `package` s `is required, please install it`

11.4.2 Syntax errors, type errors.

11.4.2.1 e_DIM. arguments submitted to function s have inconsistent dimensions. E.g., when solving a linear system, or trying to compute the determinant of a nonsquare matrix.

```
pari_err(e_DIM, const char *s)
pari_err_DIM(const char *s)
```

prints the error message: `inconsistent dimensions in` s.

11.4.2.2 e_FLAG. A flag argument is out of bounds in function s.

```
pari_err(e_FLAG, const char *s)
pari_err_FLAG(const char *s)
```

prints the error message: `invalid flag in` s.

11.4.2.3 e_NOTFUNC. Generated by the PARI evaluator; tried to use a `GEN` which is not a `t_CLOSURE` in a function call syntax (as in `f = 1; f(2);`).

```
pari_err(e_NOTFUNC, GEN fun)
```

prints the error message: `not a function in a function call`.

11.4.2.4 e_OP. Impossible operation between two objects than cannot be typecast to a sensible common domain for deeper reasons than a type mismatch, usually for arithmetic reasons. As in `O(2) + O(3)`: it is valid to add two `t_PADIC`s, provided the underlying prime is the same; so the addition is not forbidden a priori for type reasons, it only becomes so when inspecting the objects and trying to perform the operation.

```
pari_err(e_OP, const char *op, GEN x, GEN y)
pari_err_OP(const char *op, GEN x, GEN y)
```

As e_TYPE2, replacing `forbidden` by `inconsistent`.

11.4.2.5 e_PRIORITY. object o in function s contains variables whose priority is incompatible with the expected operation. E.g. `Pol([x,1], 'y)`: this raises an error because it's not possible to create a polynomial whose coefficients involve variables with higher priority than the main variable.

```
pari_err(e_PRIORITY, const char *s, GEN o, const char *op, long v)
pari_err_PRIORITY(const char *s, GEN o, const char *op, long v)
```

prints the error message: `incorrect priority in` s`, variable` v_o *op* v, were v_o is `gvar(o)`.

11.4.2.6 e_SYNTAX. Syntax error, generated by the PARI parser.

```
pari_err(e_SYNTAX, const char *msg, const char *e, const char *entry)
```

where `msg` is a complete error message, and `e` and `entry` point into the *same* character string, which is the input that was incorrectly parsed: `e` points to the character where the parser failed, and `entry` $\leq$ `e` points somewhat before.

Prints the error message: `msg`, followed by a colon, then a part of the input character string (in general `entry` itself, but an initial segment may be truncated if `e` $-$ `entry` is large); a caret points at `e`, indicating where the error took place.

11.4.2.7 e_TYPE. An argument x of function s had an unexpected type. (As in `factor("blah")`.)

```
pari_err(e_TYPE, const char *s, GEN x)
pari_err_TYPE(const char *s, GEN x)
```

prints the error message: `incorrect type in` s `(t_`x`)`, where `t_`x is the type of x.

11.4.2.8 e_TYPE2. Forbidden operation between two objects than cannot be typecast to a sensible common domain, because their types do not match up. (As in `Mod(1,2) + Pi`.)

```
pari_err(e_TYPE2, const char *op, GEN x, GEN y)
pari_err_TYPE2(const char *op, GEN x, GEN y)
```

prints the error message: `forbidden` s `t_`x *op* `t_`y, where `t_`z denotes the type of z. Here, s denotes the spelled out name of the operator $op \in \{+, *, /, \%, =\}$, e.g. *addition* for `"+"` or *assignment* for `"="`. If *op* is not in the above operator, list, it is taken to be the already spelled out name of a function, e.g. `"gcd"`, and the error message becomes `forbidden` *op* `t_`x, `t_`y.

11.4.2.9 e_VAR. polynomials x and y submitted to function s have inconsistent variables. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfinit(x^2+1)`.

```
pari_err(e_VAR, const char *s, GEN x, GEN y)
pari_err_VAR(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent variables in` s X `!=` Y, where X and Y are the names of the variables of x and y, respectively.

11.4.3 Overflows.

11.4.3.1 e_COMPONENT. Trying to access an inexistent component of a vector/matrix/list: the index is less than 1 or greater than the allowed length.

```
pari_err(e_COMPONENT, const char *f, const char *op, GEN lim, GEN x)
pari_err_COMPONENT(const char *f, const char *op, GEN lim, GEN x)
```

prints the error message: `nonexistent component in` f`: index` *op* *lim*. Special case: if f is the empty string (no meaningful public function name can be used), we ignore it and print the message: `nonexistent component: index` *op* *lim*.

11.4.3.2 e_DOMAIN. An argument x is not in the function's domain (as in `moebius(0)` or `zeta(1)`).

```
pari_err(e_DOMAIN, char *f, char *v, char *op, GEN lim, GEN x)
pari_err_DOMAIN(char *f, char *v, char *op, GEN lim, GEN x)
```

prints the error message: `domain error in` *f*`:` *v op lim*. Special case: if *op* is the empty string, we ignore *lim* and print the error message: `domain error in` *f*`:` *v* `out of range`.

11.4.3.3 e_MAXPRIME. A function using the precomputed list of prime numbers ran out of primes.

```
pari_err(e_MAXPRIME, ulong c)
pari_err_MAXPRIME(ulong c)
```

prints the error message: `not enough precomputed primes, need primelimit ~`*c* if *c* is nonzero. And simply `not enough precomputed primes` otherwise.

11.4.3.4 e_MEM. A call to `pari_malloc` or `pari_realloc` failed.

```
pari_err(e_MEM)
```

prints the error message: `not enough memory`.

11.4.3.5 e_OVERFLOW. An object in function s becomes too large to be represented within PARI's hardcoded limits. (As in `2^2^2^10` or `exp(1e100)`, which overflow in `lg` and `expo`.)

```
pari_err(e_OVERFLOW, const char *s)
pari_err_OVERFLOW(const char *s)
```

prints the error message: `overflow in` *s*.

11.4.3.6 e_PREC. Function s fails because input accuracy is too low. (As in `floor(1e100)` at default accuracy.)

```
pari_err(e_PREC, const char *s)
pari_err_PREC(const char *s)
```

prints the error message: `precision too low in` *s*.

11.4.3.7 e_STACK. The PARI stack overflows.

```
pari_err(e_STACK)
```

prints the error message: `the PARI stack overflows !` as well as some statistics concerning stack usage.

11.4.4 Errors triggered intentionally.

11.4.4.1 e_ALARM. A timeout, generated by the `alarm` function.

```
pari_err(e_ALARM, const char *fmt, ...)
```

prints the error message: *s*.

11.4.4.2 e_USER. A user error, as triggered by `error`$(g_1, \ldots, g_n)$ in GP.

```
pari_err(e_USER, GEN g)
```

prints the error message: `user error:`, then the entries of the vector g.

11.4.5 Mathematical errors.

11.4.5.1 e_CONSTPOL. An argument of function s is a constant polynomial, which does not make sense. (As in `galoisinit(Pol(1))`.)

```
pari_err(e_CONSTPOL, const char *s)
pari_err_CONSTPOL(const char *s)
```

prints the error message: `constant polynomial in` s.

11.4.5.2 e_COPRIME. Function s expected two coprime arguments, and did receive x, y which were not.

```
pari_err(e_COPRIME, const char *s, GEN x, GEN y)
pari_err_COPRIME(const char *s, GEN x, GEN y)
```

prints the error message: `elements not coprime in` s`:` x, y.

11.4.5.3 e_INV. Tried to invert a noninvertible object x.

```
pari_err(e_INV, const char *s, GEN x)
pari_err_INV(const char *s, GEN x)
```

prints the error message: `impossible inverse in` s`:` x. If $x =$ `Mod`(a, b) is a `t_INTMOD` and a is not 0 mod b, this allows to factor the modulus, as `gcd`(a, b) is a nontrivial divisor of b.

11.4.5.4 e_IRREDPOL. Function s expected an irreducible polynomial, and did not receive one. (As in `nfinit(x^2-1)`.)

```
pari_err(e_IRREDPOL, const char *s, GEN x)
pari_err_IRREDPOL(const char *s, GEN x)
```

prints the error message: `not an irreducible polynomial in` s`:` x.

11.4.5.5 e_MISC. Generic uncategorized error.

```
pari_err(e_MISC, const char *fmt, ...)
```

prints the error message: s.

11.4.5.6 e_MODULUS. moduli x and y submitted to function s are inconsistent. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfinit(t^3-2)`.

```
pari_err(e_MODULUS, const char *s, GEN x, GEN y)
pari_err_MODULUS(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent moduli in` s, then the moduli.

11.4.5.7 e_PRIME. Function s expected a prime number, and did receive p, which was not. (As in `idealprimedec(nf, 4)`.)

```
pari_err(e_PRIME, const char *s, GEN x)
pari_err_PRIME(const char *s, GEN x)
```

prints the error message: `not a prime in` s`:` x.

11.4.5.8 e_ROOTS0. An argument of function s is a zero polynomial, and we need to consider its roots. (As in `polroots(0)`.)

```
pari_err(e_ROOTS0, const char *s)
pari_err_ROOTS0(const char *s)
```

prints the error message: `zero polynomial in` s.

11.4.5.9 e_SQRTN. Tried to compute an n-th root of x, which does not exist, in function s. (As in `sqrt(Mod(-1,3))`.)

```
pari_err(e_SQRTN, GEN x)
pari_err_SQRTN(GEN x)
```

prints the error message: `not an n-th power residue in` s`:` x.

11.4.6 Miscellaneous functions.

`long name_numerr(const char *s)` return the error number corresponding to an error name. E.g. `name_numerr("e_DIM")` returns `e_DIM`.

`const char* numerr_name(long errnum)` returns the error name corresponding to an error number. E.g. `name_numerr(e_DIM)` returns `"e_DIM"`.

`char* pari_err2str(GEN err)` returns the error message that would be printed on `t_ERROR` `err`. The name is allocated on the PARI stack and must not be freed.

11.5 Hashtables.

A `hashtable`, or associative array, is a set of pairs (k, v) of keys and values. PARI implements general extensible hashtables for fast data retrieval: when creating a table, we may either choose to use the PARI stack, or `malloc` so as to be stack-independent. A hashtable is implemented as a table of linked lists, each list containing all entries sharing the same hash value. The table length is a prime number, which roughly doubles as the table overflows by gaining new entries; both the current number of entries and the threshold before the table grows are stored in the table. Finally the table remembers the functions used to hash the entries's keys and to test for equality two entries hashed to the same value.

An entry, or `hashentry`, contains

• a key/value pair (k, v), both of type `void*` for maximal flexibility,

• the hash value of the key, for the table hash function. This hash is mapped to a table index (by reduction modulo the table length), but it contains more information, and is used to bypass costly general equality tests if possible,

• a link pointer to the next entry sharing the same table cell.

```
typedef struct {
  void *key, *val;
  ulong hash; /* hash(key) */
  struct hashentry *next;
} hashentry;
```

```
typedef struct {
  ulong len; /* table length */
```

```
    hashentry **table; /* the table */
    ulong nb, maxnb; /* number of entries stored and max nb before enlarging */
    ulong pindex; /* prime index */
    ulong (*hash) (void *k); /* hash function */
    int (*eq) (void *k1, void *k2); /* equality test */
    int use_stack; /* use the PARI stack, resp. malloc */
  } hashtable;
```

```
hashtable* hash_create(size, hash, eq, use_stack)
      ulong size;
      ulong (*hash)(void*);
      int (*eq)(void*,void*);
      int use_stack;
```

creates a hashtable with enough room to contain `size` entries. The functions `hash` and `eq` compute the hash value of keys and test keys for equality, respectively. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`hashtable* hash_create_ulong(ulong size, long stack)` special case when the keys are `ulongs` with ordinary equality test.

`hashtable* hash_create_str(ulong size, long stack)` special case when the keys are character strings with string equality test (and `hash_str` hash function).

`void hash_init(hashtable *h, ulong size, ulong (*hash)(void*), int (*eq)(void*, void*), use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `void*`. The functions `eq` test keys for equality. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`void hash_init_GEN(hashtable *h, ulong size, int (*eq)(GEN, GEN), use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `GEN`. The functions `eq` test keys for equality. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`. The hash used is `hash_GEN`.

`void hash_init_ulong(hashtable *h, ulong size, use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `ulong`. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`void hash_insert(hashtable *h, void *k, void *v)` inserts (k, v) in hashtable h. No copy is made: k and v themselves are stored. The implementation does not prevent one to insert two entries with equal keys k, but which of the two is affected by later commands is undefined.

`void hash_insert2(hashtable *h, void *k, void *v, ulong hash)` as `hash_insert`, assuming `h->hash(k)` is `hash`.

`void hash_insert_long(hashtable *h, void *k, long v)` as `hash_insert` but `v` is a `long`.

`hashentry* hash_search(hashtable *h, void *k)` look for an entry with key k in h. Return it if it one exists, and `NULL` if not.

`hashentry* hash_search2(hashtable *h, void *k, ulong hash)` as `hash_search` assuming `h->hash(k)` is `hash`.

`GEN hash_haskey_GEN(hashtable *h, void *k)` returns the associate value if the key k belongs to the hash, otherwise returns `NULL`.

`int hash_haskey_long(hashtable *h, void *k, long *v)` returns 1 if the key k belongs to the hash and set v to its value, otherwise returns 0.

`hashentry * hash_select(hashtable *h, void *k, void *E, int (*select)(void *, hashentry *))` variant of `hash_search`, useful when entries with identical keys are inserted: among the entries attached to key k, return one satisfying the selection criterion (such that `select(E,e)` is nonzero), or `NULL` if none exist.

`hashentry* hash_remove(hashtable *h, void *k)` deletes an entry (k, v) with key k from h and return it. (Return `NULL` if none was found.) Only the linking structures are freed, memory attached to k and v is not reclaimed.

`hashentry* hash_remove_select(hashtable *h, void *k, void *E, int(*select)(void*, hashentry *))` a variant of `hash_remove`, useful when entries with identical keys are inserted: among the entries attached to key k, return one satisfying the selection criterion (such that `select(E,e)` is nonzero) and delete it, or `NULL` if none exist. Only the linking structures are freed, memory attached to k and v is not reclaimed.

`GEN hash_keys(hashtable *h)` return in a `t_VECSMALL` the keys stored in hashtable h.

`GEN hash_values(hashtable *h)` return in a `t_VECSMALL` the values stored in hashtable h.

`void hash_destroy(hashtable *h)` deletes the hashtable, by removing all entries.

`void hash_dbg(hashtable *h)` print statistics for hashtable h, allows to evaluate the attached hash function performance on actual data.

Some interesting hash functions are available:

`ulong hash_str(const char *s)`

`ulong hash_str_len(const char *s, long len)` hash the prefix string containing the first `len` characters (assume `strlen`$(s) \geq$ `len`).

`ulong hash_GEN(GEN x)` generic hash function.

`ulong hash_zv(GEN x)` hash a `t_VECSMALL`.

11.6 Dynamic arrays.

A **dynamic array** is a generic way to manage stacks of data that need to grow dynamically. It allocates memory using `pari_malloc`, and is independent of the PARI stack; it even works before the `pari_init` call.

11.6.1 Initialization.

To create a stack of objects of type `foo`, we proceed as follows:

```
foo *t_foo;
pari_stack s_foo;
pari_stack_init(&s_foo, sizeof(*t_foo), (void**)&t_foo);
```

Think of `s_foo` as the controlling interface, and `t_foo` as the (dynamic) array tied to it. The value of `t_foo` may be changed as you add more elements.

11.6.2 Adding elements. The following function pushes an element on the stack.

```
/* access globals t_foo and s_foo */
void push_foo(foo x)
{
  long n = pari_stack_new(&s_foo);
  t_foo[n] = x;
}
```

11.6.3 Accessing elements.

Elements are accessed naturally through the `t_foo` pointer. For example this function swaps two elements:

```
void swapfoo(long a, long b)
{
  foo x;
  if (a > s_foo.n || b > s_foo.n) pari_err_BUG("swapfoo");
  x        = t_foo[a];
  t_foo[a] = t_foo[b];
  t_foo[b] = x;
}
```

11.6.4 Stack of stacks. Changing the address of `t_foo` is not supported in general. In particular `realloc()`'ed array of stacks and stack of stacks are not supported.

11.6.5 Public interface. Let `s` be a `pari_stack` and `data` the data linked to it. The following public fields are defined:

• `s.alloc` is the number of elements allocated for `data`.

• `s.n` is the number of elements in the stack and `data[s.n-1]` is the topmost element of the stack. `s.n` can be changed as long as $0 \leq$ `s.n` $\leq$ `s.alloc` holds.

`void pari_stack_init(pari_stack *s, size_t size, void **data)` links `*s` to the data pointer `*data`, where `size` is the size of data element. The pointer `*data` is set to `NULL`, `s->n` and `s->alloc` are set to 0: the array is empty.

`void pari_stack_alloc(pari_stack *s, long nb)` makes room for `nb` more elements, i.e. makes sure that `s.alloc` $\geq$ `s.n` + `nb`, possibly reallocating `data`.

`long pari_stack_new(pari_stack *s)` increases `s.n` by one unit, possibly reallocating `data`, and returns `s.n` − 1.

Caveat. The following construction is incorrect because `stack_new` can change the value of `t_foo`:

```
t_foo[ pari_stack_new(&s_foo) ] = x;
```

`void pari_stack_delete(pari_stack *s)` frees `data` and resets the stack to the state immediately following `stack_init` (`s->n` and `s->alloc` are set to 0).

`void * pari_stack_pushp(pari_stack *s, void *u)` This function assumes that `*data` is of pointer type. Pushes the element `u` on the stack `s`.

`void ** pari_stack_base(pari_stack *s)` returns the address of `data`, typecast to a `void **`.

11.7 Vectors and Matrices.

11.7.1 Access and extract. See Section 9.3.1 and Section 9.3.2 for various useful constructors. Coefficients are accessed and set using `gel`, `gcoeff`, see Section 5.2.7. There are many internal functions to extract or manipulate subvectors or submatrices but, like the accessors above, none of them are suitable for `gerepileupto`. Worse, there are no type verification, nor bound checking, so use at your own risk.

`GEN shallowcopy(GEN x)` returns a `GEN` whose components are the components of x (no copy is made). The result may now be used to compute in place without destroying x. This is essentially equivalent to

```
GEN y = cgetg(lg(x), typ(x));
for (i = 1; i < lg(x); i++) y[i] = x[i];
return y;
```

except that `t_MAT` is treated specially since shallow copies of all columns are made. The function also works for nonrecursive types, but is useless in that case since it makes a deep copy. If x is known to be a `t_MAT`, you may call `RgM_shallowcopy` directly; if x is known not to be a `t_MAT`, you may call `leafcopy` directly.

`GEN RgM_shallowcopy(GEN x)` returns `shallowcopy(x)`, where x is a `t_MAT`.

`GEN shallowtrans(GEN x)` returns the transpose of x, *without* copying its components, i. e., it returns a `GEN` whose components are (physically) the components of x. This is the internal function underlying `gtrans`.

`GEN shallowconcat(GEN x, GEN y)` concatenate x and y, *without* copying components, i. e., it returns a `GEN` whose components are (physically) the components of x and y.

`GEN shallowconcat1(GEN x)` x must be `t_VEC`, `t_COL` or `t_LIST`, concatenate its elements from left to right. Shallow version of `gconcat1`.

`GEN shallowmatconcat(GEN v)` shallow version of `matconcat`.

`GEN shallowextract(GEN x, GEN y)` extract components of the vector or matrix x according to the selection parameter y. This is the shallow analog of `extract0(x, y, NULL)`, see `vecextract`.

`GEN shallowmatextract(GEN M, GEN l1, GEN l2)` extract components of the matrix M according to the `t_VECSMALL` $l1$ (list of lines indices) and $l2$ (list of columns indices). This is the shallow analog of `extract0(x, l1, l2)`, see `vecextract`.

`GEN RgM_minor(GEN A, long i, long j)` given a square `t_MAT` A, return the matrix with i-th row and j-th column removed.

`GEN vconcat(GEN A, GEN B)` concatenate vertically the two `t_MAT` A and B of compatible dimensions. A `NULL` pointer is accepted for an empty matrix. See `shallowconcat`.

`GEN matslice(GEN A, long a, long b, long c, long d)` returns the submatrix $A[a..b, c..d]$. Assume $a \le b$ and $c \le d$.

`GEN row(GEN A, long i)` return $A[i,]$, the i-th row of the `t_MAT` A.

`GEN row_i(GEN A, long i, long j1, long j2)` return part of the i-th row of `t_MAT` A: $A[i, j_1]$, $A[i, j_1 + 1] \dots, A[i, j_2]$. Assume $j_1 \le j_2$.

`GEN rowcopy(GEN A, long i)` return the row $A[i,]$ of the `t_MAT` A. This function is memory clean and suitable for `gerepileupto`. See `row` for the shallow equivalent.

`GEN rowslice(GEN A, long i1, long i2)` return the `t_MAT` formed by the i_1-th through i_2-th rows of `t_MAT` A. Assume $i_1 \leq i_2$.

`GEN rowsplice(GEN A, long i)` return the `t_MAT` formed from the coefficients of `t_MAT` A with j-th row removed.

`GEN rowpermute(GEN A, GEN p)`, p being a `t_VECSMALL` representing a list $[p_1, \ldots, p_n]$ of rows of `t_MAT` A, returns the matrix whose rows are $A[p_1,], \ldots, A[p_n,]$.

`GEN rowslicepermute(GEN A, GEN p, long x1, long x2)`, short for

```
rowslice(rowpermute(A,p), x1, x2)
```

(more efficient).

`GEN vecslice(GEN A, long j1, long j2)`, return $A[j_1], \ldots, A[j_2]$. If A is a `t_MAT`, these correspond to *columns* of A. The object returned has the same type as A (`t_VECSMALL`, `t_VEC`, `t_COL` or `t_MAT`). Assume $j_1 \leq j_2$ or $j_2 = j_1 - 1$ (return empty vector/matrix).

`GEN vecsplice(GEN A, long j)` return A with j-th entry removed (`t_VEC`, `t_COL`) or j-th column removed (`t_MAT`).

`GEN vecreverse(GEN A)`. Returns a `GEN` which has the same type as A (`t_VEC`, `t_COL` or `t_MAT`), and whose components are the $A[n], \ldots, A[1]$. If A is a `t_MAT`, these are the *columns* of A.

`void vecreverse_inplace(GEN A)` as `vecreverse`, but reverse A in place.

`GEN vecpermute(GEN A, GEN p)` p is a `t_VECSMALL` representing a list $[p_1, \ldots, p_n]$ of indices. Returns a `GEN` which has the same type as A (`t_VEC`, `t_COL` or `t_MAT`), and whose components are $A[p_1], \ldots, A[p_n]$. If A is a `t_MAT`, these are the *columns* of A.

`GEN vecsmallpermute(GEN A, GEN p)` as `vecpermute` when `A` is a `t_VECSMALL`.

`GEN vecslicepermute(GEN A, GEN p, long y1, long y2)` short for

```
vecslice(vecpermute(A,p), y1, y2)
```

(more efficient).

11.7.2 Componentwise operations.

The following convenience routines automate trivial loops of the form

```
for (i = 1; i < lg(a); i++) gel(v,i) = f(gel(a,i), gel(b,i))
```

for suitable f:

`GEN vecinv(GEN a)`. Given a vector a, returns the vector whose i-th component is `ginv`$(a[i])$.

`GEN vecmul(GEN a, GEN b)`. Given a and b two vectors of the same length, returns the vector whose i-th component is `gmul`$(a[i], b[i])$.

`GEN vecdiv(GEN a, GEN b)`. Given a and b two vectors of the same length, returns the vector whose i-th component is `gdiv`$(a[i], b[i])$.

`GEN vecpow(GEN a, GEN n)`. Given n a `t_INT`, returns the vector whose i-th component is $a[i]^n$.

`GEN vecmodii(GEN a, GEN b)`. Assuming a and b are two `ZV` of the same length, returns the vector whose i-th component is `modii`$(a[i], b[i])$.

`GEN vecmoduu(GEN a, GEN b)`. Assuming a and b are two `t_VECSMALL` of the same length, returns the vector whose i-th component is $a[i]$ % $b[i]$.

Note that `vecadd` or `vecsub` do not exist since `gadd` and `gsub` have the expected behavior. On the other hand, `ginv` does not accept vector types, hence `vecinv`.

11.7.3 Low-level vectors and columns functions.

These functions handle `t_VEC` as an abstract container type of `GEN`s. No specific meaning is attached to the content. They accept both `t_VEC` and `t_COL` as input, but `col` functions always return `t_COL` and `vec` functions always return `t_VEC`.

Note. All the functions below are shallow.

`GEN const_col(long n, GEN x)` returns a `t_COL` of `n` components equal to `x`.

`GEN const_vec(long n, GEN x)` returns a `t_VEC` of `n` components equal to `x`.

`int vec_isconst(GEN v)` Returns 1 if all the components of `v` are equal, else returns 0.

`void vec_setconst(GEN v, GEN x)` v a pre-existing vector. Set all its components to x.

`int vec_is1to1(GEN v)` Returns 1 if the components of `v` are pair-wise distinct, i.e. if $i \mapsto v[i]$ is a 1-to-1 mapping, else returns 0.

`GEN vec_append(GEN V, GEN s)` append `s` to the vector `V`.

`GEN vec_prepend(GEN V, GEN s)` prepend `s` to the vector `V`.

`GEN vec_shorten(GEN v, long n)` shortens the vector `v` to `n` components.

`GEN vec_lengthen(GEN v, long n)` lengthens the vector `v` to `n` components. The extra components are not initialized.

`GEN vec_insert(GEN v, long n, GEN x)` inserts x at position n in the vector v.

`GEN vec_equiv(GEN O)` given a vector of objects O, return a vector with n components where n is the number of distinct objects in O. The i-th component is a `t_VECSMALL` containing the indices of the elements in O having the same value. Applied to the image of a function evaluated on some finite set, it computes the fibers of the function.

`GEN vec_reduce(GEN O, GEN *pE)` given a vector of objects O, return the vector v (of the same type as O) of *distinct* elements of O and set a `t_VECSMALL` E with the same length as v, such that $E[i]$ is the multiplicity of object $v[i]$ in the original O. Shallow function.

11.8 Vectors of small integers.

11.8.1 t_VECSMALL.

These functions handle t_VECSMALL as an abstract container type of small signed integers. No specific meaning is attached to the content.

GEN const_vecsmall(long n, long c) returns a t_VECSMALL of n components equal to c.

GEN vec_to_vecsmall(GEN z) identical to ZV_to_zv(z).

GEN vecsmall_to_vec(GEN z) identical to zv_to_ZV(z).

GEN vecsmall_to_col(GEN z) identical to zv_to_ZC(z).

GEN vecsmall_to_vec_inplace(GEN z) apply stoi to all entries of z and set its type to t_VEC.

GEN vecsmall_copy(GEN x) makes a copy of x on the stack.

GEN vecsmall_shorten(GEN v, long n) shortens the t_VECSMALL v to n components.

GEN vecsmall_lengthen(GEN v, long n) lengthens the t_VECSMALL v to n components. The extra components are not initialized.

GEN vecsmall_indexsort(GEN x) performs an indirect sort of the components of the t_VECSMALL x and return a permutation stored in a t_VECSMALL.

void vecsmall_sort(GEN v) sorts the t_VECSMALL v in place.

GEN vecsmall_reverse(GEN v) as vecreverse for a t_VECSMALL v.

long vecsmall_max(GEN v) returns the maximum of the elements of t_VECSMALL v, assumed nonempty.

long vecsmall_indexmax(GEN v) returns the index of the largest element of t_VECSMALL v, assumed nonempty.

long vecsmall_min(GEN v) returns the minimum of the elements of t_VECSMALL v, assumed nonempty.

long vecsmall_indexmin(GEN v) returns the index of the smallest element of t_VECSMALL v, assumed nonempty.

int vecsmall_isconst(GEN v) Returns 1 if all the components of v are equal, else returns 0.

int vecsmall_is1to1(GEN v) Returns 1 if the components of v are pair-wise distinct, i.e. if $i \mapsto v[i]$ is a 1-to-1 mapping, else returns 0.

long vecsmall_isin(GEN v, long x) returns the first index i such that v[i] is equal to x. Naive search in linear time, does not assume that v is sorted.

GEN vecsmall_uniq(GEN v) given a t_VECSMALL v, return the vector of unique occurrences.

GEN vecsmall_uniq_sorted(GEN v) same as vecsmall_uniq, but assumes v sorted.

long vecsmall_duplicate(GEN v) given a t_VECSMALL v, return 0 if there is no duplicates, or the index of the first duplicate (vecsmall_duplicate([1,1]) returns 2).

long vecsmall_duplicate_sorted(GEN v) same as vecsmall_duplicate, but assume v sorted.

int vecsmall_lexcmp(GEN x, GEN y) compares two t_VECSMALL lexically.

`int vecsmall_prefixcmp(GEN x, GEN y)` truncate the longest `t_VECSMALL` to the length of the shortest and compares them lexicographically.

`GEN vecsmall_prepend(GEN V, long s)` prepend `s` to the `t_VECSMALL` `V`.

`GEN vecsmall_append(GEN V, long s)` append `s` to the `t_VECSMALL` `V`.

`GEN vecsmall_concat(GEN u, GEN v)` concat the `t_VECSMALL` `u` and `v`.

`long vecsmall_coincidence(GEN u, GEN v)` returns the numbers of indices where `u` and `v` agree.

`long vecsmall_pack(GEN v, long base, long mod)` handles the `t_VECSMALL` `v` as the digit of a number in base `base` and return this number modulo `mod`. This can be used as an hash function.

`GEN vecsmall_prod(GEN v)` given a `t_VECSMALL` `v`, return the product of its entries.

11.8.2 Vectors of `t_VECSMALL`. These functions manipulate vectors of `t_VECSMALL` (vecvecsmall).

`GEN vecvecsmall_sort(GEN x)` sorts lexicographically the components of the vector `x`.

`GEN vecvecsmall_sort_shallow(GEN x)`, shallow variant of `vecvecsmall_sort`.

`void vecvecsmall_sort_inplace(GEN x, GEN *perm)` sort lexicographically `x` in place, without copying its components. If `perm` is not `NULL`, it is set to the permutation that would sort the original `x`.

`GEN vecvecsmall_sort_uniq(GEN x)` sorts lexicographically the components of the vector `x`, removing duplicates entries.

`GEN vecvecsmall_indexsort(GEN x)` performs an indirect lexicographic sorting of the components of the vector `x` and return a permutation stored in a `t_VECSMALL`.

`long vecvecsmall_search(GEN x, GEN y)` `x` being a sorted vecvecsmall and `y` a `t_VECSMALL`, search `y` inside `x`.

`GEN vecvecsmall_max(GEN x)` returns the largest entry in all $x[i]$, assumed nonempty. Shallow function.

Chapter 12:
Functions related to the GP interpreter

12.1 Handling closures.

12.1.1 Functions to evaluate t_CLOSURE.

`void closure_disassemble(GEN C)` print the `t_CLOSURE` `C` in GP assembly format.

`GEN closure_callgenall(GEN C, long n, ...)` evaluate the `t_CLOSURE` `C` with the `n` arguments (of type `GEN`) following `n` in the function call. Assumes `C` has arity $\geq$ `n`.

`GEN closure_callgenvec(GEN C, GEN args)` evaluate the `t_CLOSURE` `C` with the arguments supplied in the vector `args`. Assumes `C` has arity $\geq$ `lg(args)` $-$ `1`.

`GEN closure_callgenvecprec(GEN C, GEN args, long prec)` as `closure_callgenvec` but set the precision locally to `prec`.

`GEN closure_callgenvecdef(GEN C, GEN args, GEN def)` evaluate the `t_CLOSURE` `C` with the arguments supplied in the vector `args`, where the `t_VECSMALL` `def` indicates which arguments are actually present. Assumes `C` has arity $\geq$ `lg(args)` $-$ `1`.

`GEN closure_callgenvecdefprec(GEN C, GEN args, GEN def, long prec)` as `closure_callgenvecdef` but set the precision locally to `prec`.

`GEN closure_callgen0prec(GEN C, long prec)` evaluate the `t_CLOSURE` `C` without arguments, but set the precision locally to `prec`.

`GEN closure_callgen1(GEN C, GEN x)` evaluate the `t_CLOSURE` `C` with argument `x`. Assumes `C` has arity ≥ 1.

`GEN closure_callgen1prec(GEN C, GEN x, long prec)` as `closure_callgen1`, but set the precision locally to `prec`.

`GEN closure_callgen2(GEN C, GEN x, GEN y)` evaluate the `t_CLOSURE` `C` with argument `x`, `y`. Assumes `C` has arity ≥ 2.

`void closure_callvoid1(GEN C, GEN x)` evaluate the `t_CLOSURE` `C` with argument `x` and discard the result. Assumes `C` has arity ≥ 1.

The following technical functions are used to evaluate *inline* closures and closures of arity 0.

The control flow statements (break, next and return) will cause the evaluation of the closure to be interrupted; this is called below a *flow change*. When that occurs, the functions below generally return `NULL`. The caller can then adopt three positions:

• raises an exception (`closure_evalnobrk`).

• passes through (by returning NULL itself).

• handles the flow change.

`GEN closure_evalgen(GEN code)` evaluates a closure and returns the result, or `NULL` if a flow change occurred.

`GEN closure_evalnobrk(GEN code)` as `closure_evalgen` but raise an exception if a flow change occurs. Meant for iterators where interrupting the closure is meaningless, e.g. `intnum` or `sumnum`.

`void closure_evalvoid(GEN code)` evaluates a closure whose return value is ignored. The caller has to deal with eventual flow changes by calling `loop_break`.

The remaining functions below are for exceptional situations:

`GEN closure_evalres(GEN code)` evaluates a closure and returns the result. The difference with `closure_evalgen` being that, if the flow end by a `return` statement, the result will be the returned value instead of `NULL`. Used by the main GP loop.

`GEN closure_evalbrk(GEN code, long *status)` as `closure_evalres` but set `status` to a nonzero value if a flow change occurred. This variant is not stack clean. Used by the break loop.

`GEN closure_trapgen(long numerr, GEN code)` evaluates closure, while trapping error `numerr`. Return `(GEN)1L` if error trapped, and the result otherwise, or `NULL` if a flow change occurred. Used by trap.

12.1.2 Functions to handle control flow changes.

`long loop_break(void)` processes an eventual flow changes inside an iterator. If this function return 1, the iterator should stop.

12.1.3 Functions to deal with lexical local variables.

Function using the prototype code '`V`' need to manually create and delete a lexical variable for each code '`V`', which will be given a number $-1, -2, \ldots$.

`void push_lex(GEN a, GEN code)` creates a new lexical variable whose initial value is a on the top of the stack. This variable get the number -1, and the number of the other variables is decreased by one unit. When the first variable of a closure is created, the argument `code` must be the closure that references this lexical variable. The argument `code` must be `NULL` for all subsequent variables (if any). (The closure contains the debugging data for the variable).

`void pop_lex(long n)` deletes the n topmost lexical variables, increasing the number of other variables by n. The argument n must match the number of variables allocated through `push_lex`.

`GEN get_lex(long vn)` get the value of the variable with number `vn`.

`void set_lex(long vn, GEN x)` set the value of the variable with number `vn`.

12.1.4 Functions returning new closures.

`GEN compile_str(const char *s)` returns the closure corresponding to the GP expression s.

`GEN closure_deriv(GEN code)` returns a closure corresponding to the numerical derivative of the closure `code`.

`GEN closure_derivn(GEN code, long n)` returns a closure corresponding to the numerical derivative of order $n > 0$ of the closure `code`.

`GEN snm_closure(entree *ep, GEN data)` Let `data` be a vector of length m, `ep` be an `entree` pointing to a C function f of arity $n + m$, returns a `t_CLOSURE` object g of arity n such that $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, gel(data, 1), ..., gel(data, m))$. If `data` is `NULL`, then $m = 0$ is assumed. Shallow function.

`GEN strtofunction(char *str)` returns a closure corresponding to the built-in or install'ed function named `str`.

`GEN strtoclosure(char *str, long n, ...)` returns a closure corresponding to the built-in or install'ed function named `str` with the n last parameters set to the n `GEN`s following n. This is analogous to `snm_closure(isentry(str), mkvecn(...))` but the latter has lower overhead since it does not copy arguments, nor does it validate inputs.

In the example code below, `agm1` is set to the function `x->agm(x,1)` and `res` is set to `agm(2,1)`.

```
GEN agm1 = strtoclosure("agm",1, gen_1);
GEN res = closure_callgen1(agm1, gen_2);
```

12.1.5 Functions used by the gp debugger (break loop). `long closure_context(long s)` restores the compilation context starting at frame `s+1`, and returns the index of the topmost frame. This allow to compile expressions in the topmost lexical scope.

`void closure_err(long level)` prints a backtrace of the last 20 stack frames, starting at frame `level`, the numbering starting at 0.

12.1.6 Standard wrappers for iterators. Two families of standard wrappers are provided to interface iterators like `intnum` or `sumnum` with GP.

12.1.6.1 Standard wrappers for inline closures. These wrappers are used to implement GP functions taking inline closures as input. The object `(GEN)E` must be an inline closure which is evaluated with the lexical variable number -1 set to x.

`GEN gp_eval(void *E, GEN x)` is used for the prototype code '`E`'.

`GEN gp_evalprec(void *E, GEN x, long prec)` as `gp_eval`, but set the precision locally to `prec`.

`long gp_evalvoid(void *E, GEN x)` is used for the prototype code '`I`'. The resulting value is discarded. Return a nonzero value if a control-flow instruction request the iterator to terminate immediately.

`long gp_evalbool(void *E, GEN x)` returns the boolean `gp_eval(E, x)` evaluates to (i.e. true iff the value is nonzero).

`GEN gp_evalupto(void *E, GEN x)` memory-safe version of `gp_eval`, `gcopy`-ing the result, when the evaluator returns components of previously allocated objects (e.g. member functions).

12.1.6.2 Standard wrappers for true closures. These wrappers are used to implement GP functions taking true closures as input.

GEN gp_call(void *E, GEN x) evaluates the closure (GEN)E on x.

GEN gp_callprec(void *E, GEN x, long prec) as gp_call, but set the precision locally to prec.

GEN gp_call2(void *E, GEN x, GEN y) evaluates the closure (GEN)E on (x, y).

long gp_callbool(void *E, GEN x) evaluates the closure (GEN)E on x, returns 1 if its result is nonzero, and 0 otherwise.

long gp_callvoid(void *E, GEN x) evaluates the closure (GEN)E on x, discarding the result. Return a nonzero value if a control-flow instruction request the iterator to terminate immediately.

12.2 Defaults.

entree* pari_is_default(const char *s) return the entree structure attached to s if it is the name of a default, NULL otherwise.

GEN setdefault(const char *s, const char *v, long flag) is the low-level function underlying default0. If s is NULL, call all default setting functions with string argument NULL and flag d_ACKNOWLEDGE. Otherwise, check whether s corresponds to a default and call the corresponding default setting function with arguments v and *flag*.

We shall describe these functions below: if v is NULL, we only look at the default value (and possibly print or return it, depending on flag); otherwise the value of the default to v, possibly after some translation work. The flag is one of

- d_INITRC called while reading the gprc: print and return gnil, possibly defer until gp actually starts.
- d_RETURN return the current value, as a t_INT if possible, as a t_STR otherwise.
- d_ACKNOWLEDGE print the current value, return gnil.
- d_SILENT print nothing, return gnil.

Low-level functions called by setdefault:

GEN sd_TeXstyle(const char *v, long flag)

GEN sd_breakloop(const char *v, long flag)

GEN sd_colors(const char *v, long flag)

GEN sd_compatible(const char *v, long flag)

GEN sd_datadir(const char *v, long flag)

GEN sd_debug(const char *v, long flag)

GEN sd_debugfiles(const char *v, long flag)

GEN sd_debugmem(const char *v, long flag)

GEN sd_echo(const char *v, long flag)

GEN sd_factor_add_primes(const char *v, long flag)

GEN sd_factor_proven(const char *v, long flag)

GEN sd_format(const char *v, long flag)

GEN sd_graphcolormap(const char *v, long flag)

GEN sd_graphcolors(const char *v, long flag)

GEN sd_help(const char *v, long flag)

GEN sd_histfile(const char *v, long flag)

GEN sd_histsize(const char *v, long flag)

GEN sd_lines(const char *v, long flag)

GEN sd_linewrap(const char *v, long flag)

GEN sd_log(const char *v, long flag)

GEN sd_logfile(const char *v, long flag)

GEN sd_nbthreads(const char *v, long flag)

GEN sd_new_galois_format(const char *v, long flag)

GEN sd_output(const char *v, long flag)

GEN sd_parisize(const char *v, long flag)

GEN sd_parisizemax(const char *v, long flag)

GEN sd_path(const char *v, long flag)

GEN sd_plothsizes(const char *v, long flag)

GEN sd_prettyprinter(const char *v, long flag)

GEN sd_primelimit(const char *v, long flag)

GEN sd_prompt(const char *v, long flag)

GEN sd_prompt_cont(const char *v, long flag)

GEN sd_psfile(const char *v, long flag) The psfile default is obsolete, don't use this function.

GEN sd_readline(const char *v, long flag)

GEN sd_realbitprecision(const char *v, long flag)

GEN sd_realprecision(const char *v, long flag)

GEN sd_recover(const char *v, long flag)

GEN sd_secure(const char *v, long flag)

GEN sd_seriesprecision(const char *v, long flag)

GEN sd_simplify(const char *v, long flag)

GEN sd_sopath(const char *v, int flag)

GEN sd_strictargs(const char *v, long flag)

GEN sd_strictmatch(const char *v, long flag)

GEN sd_timer(const char *v, long flag)

GEN sd_threadsize(const char *v, long flag)

GEN sd_threadsizemax(const char *v, long flag)

Generic functions used to implement defaults: most of the above routines are implemented in terms of the following generic ones. In all routines below

• v and flag are the arguments passed to default: v is a new value (or the empty string: no change), and flag is one of d_INITRC, d_RETURN, etc.

• s is the name of the default being changed, used to display error messages or acknowledgements.

GEN sd_toggle(const char *v, long flag, const char *s, int *ptn)

• if v is neither "0" nor "1", an error is raised using pari_err.

• ptn points to the current numerical value of the toggle (1 or 0), and is set to the new value (when v is nonempty).

For instance, here is how the timer default is implemented internally:

```
GEN
sd_timer(const char *v, long flag)
{ return sd_toggle(v,flag,"timer", &(GP_DATA->chrono)); }
```

The exact behavior and return value depends on flag:

• d_RETURN: returns the new toggle value, as a GEN.

• d_ACKNOWLEDGE: prints a message indicating the new toggle value and return gnil.

• other cases: print nothing and return gnil.

GEN sd_ulong(const char *v, long flag, const char *s, ulong *ptn, ulong Min, ulong Max, const char **msg)

• ptn points to the current numerical value of the toggle, and is set to the new value (when v is nonempty).

• Min and Max point to the minimum and maximum values allowed for the default.

• v must translate to an integer in the allowed ranger, a suffix among k/K ($\times 10^3$), m/M ($\times 10^6$), or g/G ($\times 10^9$) is allowed, but no arithmetic expression.

• msg is a [NULL]-terminated array of messages or NULL (ignored). If msg is not NULL, msg[i] contains a message attached to the value i of the default. The last entry in the msg array is used as a message attached to all subsequent ones.

The exact behavior and return value depends on flag:

• d_RETURN: returns the new value, as a GEN.

• d_ACKNOWLEDGE: prints a message indicating the new value, possibly a message attached to it via the msg argument, and return gnil.

• other cases: print nothing and return gnil.

`GEN sd_intarray(const char *v, long flag, const char *s, GEN *pz)`

• records a `t_VECSMALL` array of nonnegative integers.

• `pz` points to the current `t_VECSMALL` value, and is set to the new value (when `v` is nonempty).

The exact return value depends on `flag`:

• `d_RETURN`: returns the new value, as a `t_VEC` (converted via `zv_to_ZV`)

• `d_ACKNOWLEDGE`: prints a message indicating the new value, (as a `t_VEC`) and return `gnil`.

• other cases: print nothing and return `gnil`.

`GEN sd_string(const char *v, long flag, const char *s, char **pstr)` • `v` is subjet to environment expansion, then time expansion.

• `pstr` points to the current string value, and is set to the new value (when `v` is nonempty).

12.3 Records and Lazy vectors.

The functions in this section are used to implement `ell` structures and analogous objects, which are vectors some of whose components are initialized to dummy values, later computed on demand. We start by initializing the structure:

`GEN obj_init(long d, long n)` returns an *obj* S, a `t_VEC` with d regular components, accessed as `gel(S,1)`, ..., `gel(S,d)`; together with a record of n members, all initialized to 0. The arguments d and n must be nonnegative.

After `S = obj_init(d, n)`, the prototype of our other functions are of the form

```
GEN obj_do(GEN S, long tag, ...)
```

The first argument S holds the structure to be managed. The second argument *tag* is the index of the struct member (from 1 to n) we operate on. We recommend to define an `enum` and use descriptive names instead of hardcoded numbers. For instance, if $n = 3$, after defining

```
enum { TAG_p = 1, TAG_list, TAG_data };
```

one may use `TAG_list` or 2 indifferently as a tag. The former being preferred, of course.

Technical note. In the current implementation, S is a `t_VEC` with $d+1$ entries. The first d components are ordinary `t_GEN` entries, which you can read or assign to in the customary way. But the last component `gel(S, d + 1)`, a `t_VEC` of length n initialized to `zerovec`(n), must be handled in a special way: you should never access or modify its components directly, only through the API we are about to describe. Indeed, its entries are meant to contain dynamic data, which will be stored, retrieved and replaced (for instance by a value computed to a higher accuracy), while interacting safely with intermediate `gerepile` calls. This mechanism allows to simulate C `struct`s, in a simpler way than with general hashtables, while remaining compatible with the GP language, which knows neither structs nor hashtables. It also serialize the structure in an ordinary `GEN`, which facilitates copies and garbage collection (use `gcopy` or `gerepile`), rather than having to deal with individual components of actual C `struct`s.

`GEN obj_reinit(GEN S)` make a shallow copy of S, re-initializing all dynamic components. This allows "forking" a lazy vector while avoiding both a memory leak, and storing pointers to the same data in different objects (with risks of a double free later).

`GEN obj_check(GEN S, long tag)` if the *tag*-component in S is non empty, return it. Otherwise return `NULL`. The `t_INT` 0 (initial value) is used as a sentinel to indicated an empty component.

`GEN obj_insert(GEN S, long tag, GEN O)` insert (a clone of) O as *tag*-component of S. Any previous value is deleted, and data pointing to it become invalid.

`GEN obj_insert_shallow(GEN S, long K, GEN O)` as `obj_insert`, inserting O as-is, not via a clone.

`GEN obj_checkbuild(GEN S, long tag, GEN (*build)(GEN))` if the *tag*-component of S is non empty, return it. Otherwise insert (a clone of) `build(S)` as *tag*-component in S, and return it.

`GEN obj_checkbuild_padicprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of S is non empty *and* has relative p-adic precision $\geq$ `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S, and return it.

`GEN obj_checkbuild_realprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of S is non empty *and* satisfies `gprecision` $\geq$ `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S, and return it.

`GEN obj_checkbuild_prec(GEN S, long tag, GEN (*build)(GEN, long), GEN (*gpr)(GEN), long prec)` if the *tag*-component of S is non empty *and* has precision `gpr`$(x) \geq$ `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S, and return it.

`void obj_free(GEN S)` destroys all clones stored in the n tagged components, and replace them by the initial value 0. The regular entries of S are unaffected, and S remains a valid object. This is used to avoid memory leaks.

Chapter 13:
Algebraic Number Theory

13.1 General Number Fields.

13.1.1 Number field types.

None of the following routines thoroughly check their input: they distinguish between *bona fide* structures as output by PARI routines, but designing perverse data will easily fool them. To give an example, a square matrix will be interpreted as an ideal even though the **Z**-module generated by its columns may not be an $\mathbf{Z}_K$-module (i.e. the expensive `nfisideal` routine will *not* be called).

`long nftyp(GEN x)`. Returns the type of number field structure stored in `x`, `typ_NF`, `typ_BNF`, or `typ_BNR`. Other answers are possible, meaning `x` is not a number field structure.

`GEN get_nf(GEN x, long *t)`. Extract an *nf* structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_bnf(GEN x, long *t)`. Extract a `bnf` structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_nfpol(GEN x, GEN *nf)` try to extract an *nf* structure from `x`, and sets `*nf` to `NULL` (failure) or to the *nf*. Returns the (monic, integral) polynomial defining the field.

`GEN get_bnfpol(GEN x, GEN *bnf, GEN *nf)` try to extract a *bnf* and an *nf* structure from `x`, and sets `*bnf` and `*nf` to `NULL` (failure) or to the corresponding structure. Returns the (monic, integral) polynomial defining the field.

`GEN checknf(GEN x)` if an *nf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_nf` is often more flexible.

`GEN checkbnf(GEN x)` if an *bnf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_bnf` is often more flexible.

`GEN checkbnf_i(GEN bnf)` same as `checkbnf` but return `NULL` instead of raising an exception.

`void checkbnr(GEN bnr)` Raise an exception if the argument is not a *bnr* structure.

`GEN checkbnr_i(GEN bnr)` same as `checkbnr` but returns the *bnr* or `NULL` instead of raising an exception.

`GEN checknf_i(GEN nf)` same as `checknf` but return `NULL` instead of raising an exception.

`void checkrnf(GEN rnf)` Raise an exception if the argument is not an *rnf* structure.

`int checkrnf_i(GEN rnf)` same as `checkrnf` but return 0 on failure and 1 on success.

`void checkbid(GEN bid)` Raise an exception if the argument is not a *bid* structure.

`GEN checkbid_i(GEN bid)` same as `checkbid` but return `NULL` instead of raising an exception and return `bid` on success.

`GEN checkznstar_i(GEN G)` return G if it is a *znstar*; else return `NULL` on failure.

`GEN checkgal(GEN x)` if a *galoisinit* structure can be extracted from `x`, return it; otherwise raise an exception.

`void checksqmat(GEN x, long N)` check whether `x` is a square matrix of dimension `N`. May be used to check for ideals if `N` is the field degree.

`void checkprid(GEN pr)` Raise an exception if the argument is not a prime ideal structure.

`int checkprid_i(GEN pr)` same as `checkprid` but return 0 instead of raising an exception and return 1 on success.

`int is_nf_factor(GEN F)` return 1 if F is an ideal factorization and 0 otherwise.

`int is_nf_extfactor(GEN F)` return 1 if F is an extended ideal factorization (allowing 0 or negative exponents) and 0 otherwise.

`int RgV_is_prV(GEN v)` returns 1 if the vector v contains only prime ideals and 0 otherwise.

`GEN get_prid(GEN ideal)` return the underlying prime ideal structure if one can be extracted from `ideal` (ideal or extended ideal), and return `NULL` otherwise.

`void checkabgrp(GEN v)` Raise an exception if the argument is not an abelian group structure, i.e. a `t_VEC` with either 2 or 3 entries: $[N, cyc]$ or $[N, cyc, gen]$.

`GEN abgrp_get_no(GEN x)` extract the cardinality N from an abelian group structure.

`GEN abgrp_get_cyc(GEN x)` extract the elementary divisors *cyc* from an abelian group structure.

`GEN abgrp_get_gen(GEN x)` extract the generators *gen* from an abelian group structure.

`GEN cyc_get_expo(GEN cyc)` return the exponent of the group with structure `cyc`; 0 for an infinite group.

`void checkmodpr(GEN modpr)` Raise an exception if the argument is not a `modpr` structure (from `nfmodprinit`).

`GEN get_modpr(GEN x)` return x if it is a `modpr` structure and `NULL` otherwise.

`GEN checknfelt_mod(GEN nf, GEN x, const char *s)` given an *nf* structure `nf` and a `t_POLMOD` `x`, return the attached polynomial representative (shallow) if `x` and `nf` are compatible. Raise an exception otherwise. Set s to the name of the caller for a meaningful error message.

`int check_ZKmodule_i(GEN x)` return 1 if x looks like a projective $\mathbf{Z}_K$-module, i.e., a pair $[A, I]$ where A is a matrix and I is a list of ideals and A has as many columns as I has elements. Or possibly a longer list $[A, I, \ldots]$ such as the output of `rnfpseudobasis`. Otherwise return 0.

`void check_ZKmodule(GEN x, const char *s)` raise an exception unless x is recognized as a projective $\mathbf{Z}_K$-module. Set s to the name of the caller for a meaningful error message.

`long idealtyp(GEN *ideal, GEN *fa)` The input is `ideal`, a pointer to an ideal or extended ideal; returns the type of the underlying ideal among `id_PRINCIPAL` (a number field element), `id_PRIME` (a prime ideal) `id_MAT` (an ideal in matrix form).

As a first side effect, `*ideal` is set to the underlying ideal, possibly simplified (for instance the zero ideal represented by an empty matrix is replaced by `gen_0`).

If `fa` is not `NULL`, then `*fa` is set to the extended part in the input: either `NULL` (regular ideal) or the extended part of an extended ideal.

13.1.2 Extracting info from a `nf` structure.

These functions expect a true *nf* argument attached to a number field $K = \mathbf{Q}[x]/(T)$, e.g. a *bnf* will not work. Let $n = [K : \mathbf{Q}]$ be the field degree.

`GEN nf_get_pol(GEN nf)` returns the polynomial T (monic, in $\mathbf{Z}[x]$).

`long nf_get_varn(GEN nf)` returns the variable number of the number field defining polynomial.

`long nf_get_r1(GEN nf)` returns the number of real places r_1.

`long nf_get_r2(GEN nf)` returns the number of complex places r_2.

`void nf_get_sign(GEN nf, long *r1, long *r2)` sets r_1 and r_2 to the number of real and complex places respectively. Note that $r_1 + 2r_2$ is the field degree.

`long nf_get_degree(GEN nf)` returns the number field degree, $n = r_1 + 2r_2$.

`GEN nf_get_disc(GEN nf)` returns the field discriminant.

`GEN nf_get_index(GEN nf)` returns the index of T, i.e. the index of the order generated by the power basis $(1, x, \ldots, x^{n-1})$ in the maximal order of K.

`GEN nf_get_zk(GEN nf)` returns a basis $(w_1, w_2, \ldots, w_n)$ for the maximal order of K. Those are polynomials in $\mathbf{Q}[x]$ of degree $< n$; it is guaranteed that $w_1 = 1$.

`GEN nf_get_zkden(GEN nf)` returns the denominator of `nf_get_zk`, as a positive `t_INT`.

`GEN nf_get_zkprimpart(GEN nf)` returns `nf_get_zk` times its denominator.

`GEN nf_get_invzk(GEN nf)` returns the matrix $(m_{i,j}) \in M_n(\mathbf{Z})$ giving the power basis (x^i) in terms of the (w_j), i.e such that $x^{j-1} = \sum_{i=1}^n m_{i,j} w_i$ for all $1 \leq j \leq n$; since $w_1 = 1 = x^0$, we have $m_{i,1} = \delta_{i,1}$ for all i. The conversion functions in the `algtobasis` family essentially amount to a left multiplication by this matrix.

`GEN nf_get_roots(GEN nf)` returns the r_1 real roots of the polynomial defining the number fields: first the r_1 real roots (as `t_REAL`s), then the r_2 representatives of the pairs of complex conjugates.

`GEN nf_get_allroots(GEN nf)` returns all the complex roots of T: first the r_1 real roots (as `t_REAL`s), then the r_2 pairs of complex conjugates.

`GEN nf_get_M(GEN nf)` returns the $(r_1 + r_2) \times n$ matrix M giving the embeddings of K: $M[i, j]$ contains $w_j(\alpha_i)$, where α_i is the i-th element of `nf_get_roots(nf)`. In particular, if v is an n-th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i] w_i$ of K, then `RgM_RgC_mul(M,v)` represents the embeddings of v.

`GEN nf_get_G(GEN nf)` returns a $n \times n$ real matrix G such that $Gv \cdot Gv = T_2(v)$, where v is an n-th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i] w_i$ of K and T_2 is the standard Euclidean form on $K \otimes \mathbf{R}$, i.e. $T_2(v) = \sum_\sigma |\sigma(v)|^2$, where σ runs through all n complex embeddings of K.

`GEN nf_get_roundG(GEN nf)` returns a rescaled version of G, rounded to nearest integers, specifically `RM_round_maxrank`(G).

`GEN nf_get_ramified_primes(GEN nf)` returns the vector of ramified primes.

`GEN nf_get_Tr(GEN nf)` returns the matrix of the Trace quadratic form on the basis $(w_1, \ldots, w_n)$: its (i, j) entry is $\mathrm{Tr} w_i w_j$.

`GEN nf_get_diff(GEN nf)` returns the primitive part of the inverse of the above Trace matrix.

`long nf_get_prec(GEN nf)` returns the precision (in words) to which the *nf* was computed.

13.1.3 Extracting info from a `bnf` structure.

These functions expect a true *bnf* argument, e.g. a *bnr* will not work.

`GEN bnf_get_nf(GEN bnf)` returns the underlying *nf*.

`GEN bnf_get_clgp(GEN bnf)` returns the class group in *bnf*, which is a 3-component vector $[h, cyc, gen]$.

`GEN bnf_get_cyc(GEN bnf)` returns the elementary divisors of the class group (cyclic components) $[d_1, \ldots, d_k]$, where $d_k \mid \ldots \mid d_1$.

`GEN bnf_get_gen(GEN bnf)` returns the generators $[g_1, \ldots, g_k]$ of the class group. Each g_i has order d_i, and the full module of relations between the g_i is generated by the $d_i g_i = 0$.

`GEN bnf_get_no(GEN bnf)` returns the class number.

`GEN bnf_get_reg(GEN bnf)` returns the regulator.

`GEN bnf_get_logfu(GEN bnf)` returns (complex floating point approximations to) the logarithms of the complex embeddings of our system of fundamental units.

`GEN bnf_get_fu(GEN bnf)` returns the fundamental units. Raise an error if the *bnf* does not contain units in algebraic form.

`GEN bnf_get_fu_nocheck(GEN bnf)` as `bnf_get_fu` without checking whether units are present. Do not use this unless you initialize the *bnf* yourself!

`GEN bnf_get_tuU(GEN bnf)` returns a generator of the torsion part of $\mathbf{Z}_K^*$.

`long bnf_get_tuN(GEN bnf)` returns the order of the torsion part of $\mathbf{Z}_K^*$, i.e. the number of roots of unity in K.

`GEN bnf_get_sunits(GEN bnf)` allows access to the algebraic data stored by `bnfinit(,1)`. The function returns `NULL` unless the `bnf` was initialized by `bnfinit(,1)`, else a vector $[X, U, E, \texttt{lim}]$ where

• X is a vector of rational primes and algebraic integers all of whose prime divisors have norm less than `lim`,

• U is a matrix of exponents whose columns yield the fundamental units `bnf.fu`. More precisely,

$$\texttt{bnf.fu}[j] = \prod_i X[i]^{U[i,j]}.$$

• G is a matrix of exponents whose columns yield the generators of principal ideals attached to the HNF of the `bnf` relation matrix between the maximal ideals of norm less `lim` (that generate the class group under GRH). More precisely, `bnf[5]` contains the prime factor base P (its first r elements being independant class group generators), `bnf[1]` contains a matrix W in HNF in $M_r(\mathbf{Z})$ and `bnf[2]`, contains a matrix B in $M_{r\times c}(\mathbf{Z})$. We define algebraic numbers e_j for $j \le r + c$ such that

$$\prod_{\texttt{i}\le\texttt{r}} \texttt{P}_\texttt{i}^{\texttt{W[i,j]}} = (e_j), \quad j \le r$$

$$P_j \prod_{\texttt{i}\le\texttt{r}} \texttt{P}_\texttt{i}^{\texttt{B[i,j]}} = (e_j), j > r$$

Then $e_j = \prod_i X[i]^{E[i,j]}$.

`GEN bnf_has_fu(GEN bnf)` return fundamental units in expanded form if `bnf` contains them. Else return `NULL`.

`GEN bnf_compactfu(GEN bnf)` return fundamental units as a vector of algebraic numbers in compact form if `bnf` contains them. Else return `NULL`.

`GEN bnf_compactfu_mat(GEN bnf)` as a pair (X, U), where X is a vector of S-units and U is a matrix with integer entries (without 0 rows), see `bnf_get_sunits`, if `bnf` contains them. Else return `NULL`.

13.1.4 Extracting info from a `bnr` structure.

These functions expect a true *bnr* argument.

`GEN bnr_get_bnf(GEN bnr)` returns the underlying *bnf*.

`GEN bnr_get_nf(GEN bnr)` returns the underlying *nf*.

`GEN bnr_get_clgp(GEN bnr)` returns the ray class group.

`GEN bnr_get_no(GEN bnr)` returns the ray class number.

`GEN bnr_get_cyc(GEN bnr)` returns the elementary divisors of the ray class group (cyclic components) $[d_1, \ldots, d_k]$, where $d_k \mid \ldots \mid d_1$.

`GEN bnr_get_gen(GEN bnr)` returns the generators $[g_1, \ldots, g_k]$ of the ray class group. Each g_i has order d_i, and the full module of relations between the g_i is generated by the $d_i g_i = 0$. Raise a generic error if the *bnr* does not contain the ray class group generators.

`GEN bnr_get_gen_nocheck(GEN bnr)` as `bnr_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bnr* yourself!

`GEN bnr_get_bid(GEN bnr)` returns the *bid* attached to the *bnr* modulus.

`GEN bnr_get_mod(GEN bnr)` returns the modulus attached to the *bnr*.

13.1.5 Extracting info from an `rnf` structure.

These functions expect a true *rnf* argument, attached to an extension L/K, $K = \mathbf{Q}[y]/(T)$, $L = K[x]/(P)$.

`long rnf_get_degree(GEN rnf)` returns the *relative* degree $[L : K]$.

`long rnf_get_absdegree(GEN rnf)` returns the absolute degree $[L : \mathbf{Q}]$.

`long rnf_get_nfdegree(GEN rnf)` returns the degree of the base field $[K : \mathbf{Q}]$.

`GEN rnf_get_nf(GEN rnf)` returns the base field K, an *nf* structure.

`GEN rnf_get_nfpol(GEN rnf)` returns the polynomial T defining the base field K.

`long rnf_get_nfvarn(GEN rnf)` returns the variable y attached to the base field K.

`GEN rnf_get_nfzk(GEN rnf)` returns the integer basis of the base field K.

`GEN rnf_get_pol(GEN rnf)` returns the relative polynomial defining L/K.

`long rnf_get_varn(GEN rnf)` returns the variable x attached to L.

`GEN rnf_get_zk(GEN nf)` returns the relative integer basis generating $\mathbf{Z}_L$ as a $\mathbf{Z}_K$-module, as a pseudo-matrix (A, I) in HNF.

`GEN rnf_get_disc(GEN rnf)` is the output $[\mathfrak{d}, s]$ of `rnfdisc`.

`GEN rnf_get_ramified_primes(GEN rnf)` returns the vector of rational primes below ramified primes in the relative extension, i.e. all prime numbers appearing in the factorization of

```
idealnorm(rnf_get_nf(rnf), rnf_get_disc(rnf));
```

`GEN rnf_get_idealdisc(GEN rnf)` is the ideal discriminant $\mathfrak{d}$ from `rnfdisc`.

`GEN rnf_get_index(GEN rnf)` is the index ideal $\mathfrak{f}$

`GEN rnf_get_polabs(GEN rnf)` returns an absolute polynomial defining $L/\mathbf{Q}$.

`GEN rnf_get_alpha(GEN rnf)` a root α of the polynomial defining the base field, modulo `polabs` (cf. `rnfequation`)

`GEN rnf_get_k(GEN rnf)` a small integer k such that $\theta = \beta + k\alpha$ is a root of `polabs`, where β is a root of `pol` and α a root of the polynomial defining the base field, as in `rnf_get_alpha` (cf. also `rnfequation`).

`GEN rnf_get_invzk(GEN rnf)` contains A^{-1}, where (A, I) is the chosen pseudo-basis for $\mathbf{Z}_L$ over $\mathbf{Z}_K$.

`GEN rnf_get_map(GEN rnf)` returns technical data attached to the map $K \to L$. Currently, this contains data from `rnfequation`, as well as the polynomials T and P.

13.1.6 Extracting info from a `bid` structure.

These functions expect a true *bid* argument, attached to a modulus $I = I_0 I_\infty$ in a number field K.

`GEN bid_get_mod(GEN bid)` returns the modulus attached to the *bid*.

`GEN bid_get_grp(GEN bid)` returns the abelian group attached to $(\mathbf{Z}_K/I)^*$.

`GEN bid_get_ideal(GEN bid)` return the finite part I_0 of the *bid* modulus (an integer ideal).

`GEN bid_get_arch(GEN bid)` return the Archimedean part I_∞ of the *bid* modulus as a vector of real places in vec01 format, see Section 13.1.20.

`GEN bid_get_archp(GEN bid)` return the Archimedean part I_∞ of the *bid* modulus, as a vector of real places in indices format see Section 13.1.20.

`GEN bid_get_fact(GEN bid)` returns the ideal factorization $I_0 = \prod_i \mathfrak{p}_i^{e_i}$.

`GEN bid_get_fact2(GEN bid)` as `bid_get_fact` with all factors $\mathfrak{p}^1$ with $\mathfrak{p}$ of norm 2 removed from the factorization. (They play no role in the structure of $(\mathbf{Z}_K/I)^*$, except that the generators must be made coprime to them.)

`bid_get_ideal`(*bid*), via `idealfactor`.

`GEN bid_get_no(GEN bid)` returns the cardinality of the group $(\mathbf{Z}_K/I)^*$.

`GEN bid_get_cyc(GEN bid)` returns the elementary divisors of the group $(\mathbf{Z}_K/I)^*$ (cyclic components) $[d_1, \ldots, d_k]$, where $d_k \mid \ldots \mid d_1$.

`GEN bid_get_gen(GEN bid)` returns the generators of $(\mathbf{Z}_K/I)^*$ contained in *bid*. Raise a generic error if *bid* does not contain generators.

GEN bid_get_gen_nocheck(GEN bid) as bid_get_gen without checking whether generators are present. Do not use this unless you initialize the *bid* yourself!

GEN bid_get_sprk(GEN bid) return a list of structures attached to the $(\mathbf{Z}_K/\mathfrak{p}^e)^*$ where $\mathfrak{p}^e$ divides I_0 exactly.

GEN bid_get_sarch(GEN bid) return the structure attached to $(\mathbf{Z}_K/I_\infty)^*$, by nfarchstar.

GEN bid_get_U(GEN bid) return the matrix with integral coefficients relating the local generators (from chinese remainders) to the global SNF generators (*bid*.gen).

13.1.7 Extracting info from a znstar structure.

These functions expect an argument G as returned by znstar0(N, 1), attached to a positive N and the abelian group $(\mathbf{Z}/N\mathbf{Z})^*$. Let (g_i) be the SNF generators, where g_i has order d_i; we call (g_i') the (canonical) Conrey generators, where g_i' has order d_i'. Both sets of generators have the same cardinality.

GEN znstar_get_N(GEN bid) return N.

GEN znstar_get_faN(GEN G) return the factorization factor(N), $N = \prod_j p_j^{e_j}$.

GEN znstar_get_pe(GEN G) return the vector of primary factors $(p_j^{e_j})$.

GEN znstar_get_no(GEN G) the cardinality $\phi(N)$ of G.

GEN znstar_get_cyc(GEN G) elementary divisors (d_i) of $(\mathbf{Z}/N\mathbf{Z})^*$.

GEN znstar_get_gen(GEN G) SNF generators divisors (g_i) of $(\mathbf{Z}/N\mathbf{Z})^*$.

GEN znstar_get_conreycyc(GEN G) orders (d_i') of Conrey generators.

GEN znstar_get_conreygen(GEN G) Conrey generators (g_i').

GEN znstar_get_U(GEN G) a square matrix U such that $(g_i) = U(g_i')$.

GEN znstar_get_Ui(GEN G) a square matrix U' such that $U'(g_i) = (g_i')$. In general, UU' will not be the identity.

13.1.8 Inserting info in a number field structure.

If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

These functions expect a bnf argument:

GEN bnf_build_cycgen(GEN bnf) the *bnf* contains generators $[g_1, \ldots, g_k]$ of the class group, each with order d_i. Then $g_i^{d_i} = (x_i)$ is a principal ideal. This function returns the x_i as a factorization matrix (famat) giving the element in factored form as a product of S-units.

GEN bnf_build_matalpha(GEN bnf) the class group was computed using a factorbase S of prime ideals $\mathfrak{p}_i$, $i \leq r$. They satisfy relations of the form $\prod_j \mathfrak{p}_i^{e_{i,j}} = (\alpha_j)$, where the $e_{i,j}$ are given by the matrices *bnf*[1] (W, singling out a minimal set of generators in S) and *bnf*[2] (B, expressing the rest of S in terms of the singled out generators). This function returns the α_j in factored form as a product of S-units.

GEN bnf_build_units(GEN bnf) returns a minimal set of generators for the unit group in expanded form. The first element is a torsion unit, the others have infinite order. This expands units in compact form contained in a `bnf` from `bnfinit`$(,1)$ and may be *very* expensive if the units are huge.

`GEN bnf_build_cheapfu(GEN bnf)` as `bnf_build_units` but only expand units in compact form if the computation is inexpensive (a few seconds). Return `NULL` otherwise.

These functions expect a `rnf` argument:

`GEN rnf_build_nfabs(GEN rnf, long prec)` given a *rnf* structure attached to L/K, (compute and) return an *nf* structure attached to L at precision `prec`.

`void rnfcomplete(GEN rnf)` as `rnf_build_nfabs` using the precision of K for `prec`.

`GEN rnf_zkabs(GEN rnf)` returns a $\mathbf{Z}$-basis in HNF for $\mathbf{Z}_L$ as a pair $[T,v]$, where T is `rnf_get_polabs`(*rnf*) and v a vector of elements lifted from $\mathbf{Q}[X]/(T)$. Note that the function `rnf_build_nfabs` essentially applies `nfinit` to the output of this function.

13.1.9 Increasing accuracy.

`GEN nfnewprec(GEN x, long prec)`. Raise an exception if `x` is not a number field structure (*nf*, *bnf* or *bnr*). Otherwise, sets its accuracy to `prec` and return the new structure. This is mostly useful with `prec` larger than the accuracy to which `x` was computed, but it is also possible to decrease the accuracy of `x` (truncating relevant components, which may speed up later computations). This routine may modify the original `x` (see below).

This routine is straightforward for *nf* structures, but for the other ones, it requires all principal ideals corresponding to the *bnf* relations in algebraic form (they are originally only available via floating point approximations). This in turn requires many calls to `bnfisprincipal0`, which is often slow, and may fail if the initial accuracy was too low. In this case, the routine will not actually fail but recomputes a *bnf* from scratch!

Since this process may be very expensive, the corresponding data is cached (as a *clone*) in the *original* `x` so that later precision increases become very fast. In particular, the copy returned by `nfnewprec` also contains this additional data.

`GEN bnfnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnf* structure form `x` before increasing its accuracy, and returns only the latter.

`GEN bnrnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnr* structure form `x` before increasing its accuracy, and returns only the latter.

`GEN nfnewprec_shallow(GEN nf, long prec)`

`GEN bnfnewprec_shallow(GEN bnf, long prec)`

`GEN bnrnewprec_shallow(GEN bnr, long prec)` Shallow functions underlying the above, except that the first argument must now have the corresponding number field type. I.e. one cannot call `nfnewprec_shallow(nf, prec)` if `nf` is actually a *bnf*.

13.1.10 Number field arithmetic. The number field $K = \mathbf{Q}[X]/(T)$ is represented by an `nf` (or `bnf` or `bnr` structure). An algebraic number belonging to K is given as

• a `t_INT`, `t_FRAC` or `t_POL` (implicitly modulo T), or

• a `t_POLMOD` (modulo T), or

• a `t_COL` `v` of dimension $N = [K : \mathbf{Q}]$, representing the element in terms of the computed integral basis (e_i), as

```
sum(i = 1, N, v[i] * nf.zk[i])
```

The preferred forms are `t_INT` and `t_COL` of `t_INT`. Routines can handle denominators but it is much more efficient to remove denominators first (`Q_remove_denom`) and take them into account at the end.

Safe routines. The following routines do not assume that their `nf` argument is a true *nf* (it can be any number field type, e.g. a *bnf*), and accept number field elements in all the above forms. They return their result in `t_COL` form.

`GEN nfadd(GEN nf, GEN x, GEN y)` returns $x + y$.

`GEN nfsub(GEN nf, GEN x, GEN y)` returns $x - y$.

`GEN nfdiv(GEN nf, GEN x, GEN y)` returns x/y.

`GEN nfinv(GEN nf, GEN x)` returns x^{-1}.

`GEN nfmul(GEN nf, GEN x, GEN y)` returns xy.

`GEN nfpow(GEN nf, GEN x, GEN k)` returns x^k, k is in $\mathbf{Z}$.

`GEN nfpow_u(GEN nf, GEN x, ulong k)` returns x^k, $k \geq 0$; the argument `nf` is a true *nf* structure.

`GEN nfsqr(GEN nf, GEN x)` returns x^2.

`long nfval(GEN nf, GEN x, GEN pr)` returns the valuation of x at the maximal ideal $\mathfrak{p}$ attached to the *prid* `pr`. Returns `LONG_MAX` if x is 0.

`GEN nfnorm(GEN nf, GEN x)` absolute norm of x.

`GEN nftrace(GEN nf, GEN x)` absolute trace of x.

`GEN nfpoleval(GEN nf, GEN pol, GEN a)` evaluate the `t_POL` `pol` (with coefficients in `nf`) on the algebraic number a (also in nf).

`GEN FpX_FpC_nfpoleval(GEN nf, GEN pol, GEN a, GEN p)` evaluate the `FpX` `pol` on the algebraic number a (also in nf).

The following three functions implement trivial functionality akin to Euclidean division for which we currently have no real use. Of course, even if the number field is actually Euclidean, these do not in general implement a true Euclidean division.

`GEN nfdiveuc(GEN nf, GEN a, GEN b)` returns the algebraic integer closest to x/y. Functionally identical to `ground( nfdiv(nf,x,y) )`.

`GEN nfdivrem(GEN nf, GEN a, GEN b)` returns the vector $[q, r]$, where

```
q = nfdiveuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));    \\ or r = nfmod(nf,a,b);
```

`GEN nfmod(GEN nf, GEN a, GEN b)` returns r such that

```
q = nfdiveuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));
```

`GEN nf_to_scalar_or_basis(GEN nf, GEN x)` let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its basis representation (`nfalgtobasis`). Shallow function.

`GEN nf_to_scalar_or_alg(GEN nf, GEN x)` let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its lifted `t_POLMOD` representation (lifted `nfbasistoalg`). Shallow function.

`GEN nfV_to_scalar_or_alg(GEN nf, GEN v)` aplly `nf_to_scalar_or_alg` to all components of vector v.

`GEN RgX_to_nfX(GEN nf, GEN x)` let x be a `t_POL` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new polynomial. Shallow function.

`GEN RgM_to_nfM(GEN nf, GEN x)` let x be a `t_MAT` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new matrix. Shallow function.

`GEN RgC_to_nfC(GEN nf, GEN x)` let x be a `t_COL` or `t_VEC` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new `t_COL`. Shallow function.

`GEN nfX_to_monic(GEN nf, GEN T, GEN *pL)` given a nonzero `t_POL` T with coefficients in *nf*, return a monic polynomial f with integral coefficients such that $f(x) = CT(x/L)$ for some integral L and some C in *nf*. The function allows coefficients in basis form; if $L \neq 1$, it will return them in algebraic form. If `pL` is not `NULL`, `*pL` is set to L. Shallow function.

Unsafe routines. The following routines assume that their `nf` argument is a true *nf* (e.g. a *bnf* is not allowed) and their argument are restricted in various ways, see the precise description below.

`GEN nfX_disc(GEN nf, GEN A)` given an *nf* structure attached to a number field K with main variable Y (`nf_get_varn(nf)`), a `t_POL` $A \in K[X]$ given as a lift in $\mathbf{Q}[X,Y]$ (implicitly modulo `nf_get_pol(nf)`, return the discriminant of A as a `t_POL` in $\mathbf{Q}[Y]$ (representing an element of K).

`GEN nfX_resultant(GEN nf, GEN A, GEN B)` analogous to `nfX_disc`, $A, B \in \mathbf{Q}[X,Y]$; return the resultant of A and B with respect to X as a `t_POL` in $\mathbf{Q}[Y]$ (representing an element of K).

`GEN nfinvmodideal(GEN nf, GEN x, GEN A)` given an algebraic integer x and a nonzero integral ideal A in HNF, returns a y such that $xy \equiv 1$ modulo A.

`GEN nfpowmodideal(GEN nf, GEN x, GEN n, GEN ideal)` given an algebraic integer x, an integer n, and a nonzero integral ideal A in HNF, returns an algebraic integer congruent to x^n modulo A.

`GEN nfmuli(GEN nf, GEN x, GEN y)` returns $x \times y$ assuming that both x and y are either `t_INT`s or `ZV`s of the correct dimension. The argument `nf` is a true *nf* structure.

`GEN nfsqri(GEN nf, GEN x)` returns x^2 assuming that x is a `t_INT` or a `ZV` of the correct dimension. The argument `nf` is a true *nf* structure.

`GEN nfC_nf_mul(GEN nf, GEN v, GEN x)` given a `t_VEC` or `t_COL` v of elements of K in `t_INT`, `t_FRAC` or `t_COL` form, multiply it by the element x (arbitrary form). This is faster than multiplying coordinatewise since pre-computations related to x (computing the multiplication table) are done only once. The components of the result are in most cases `t_COL`s but are allowed to be `t_INT`s or `t_FRAC`s. Shallow function.

`GEN nfC_multable_mul(GEN v, GEN mx)` same as `nfC_nf_mul`, where the argument x is replaced by its multiplication table `mx`.

`GEN zkC_multable_mul(GEN v, GEN x)` same as `nfC_nf_mul`, where v is a vector of algebraic integers, x is an algebraic integer, and x is replaced by `zk_multable`(x).

`GEN zk_multable(GEN nf, GEN x)` given a `ZC` x (implicitly representing an algebraic integer), returns the `ZM` giving the multiplication table by x. Shallow function (the first column of the result points to the same data as x).

`GEN zk_inv(GEN nf, GEN x)` given a `ZC` x (implicitly representing an algebraic integer), returns the `QC` giving the inverse x^{-1}. Return `NULL` if x is 0. Not memory clean but safe for `gerepileupto`.

`GEN zkmultable_inv(GEN mx)` as `zk_inv`, where the argument given is `zk_multable`(x).

`GEN zkmultable_capZ(GEN mx)` given a nonzero *zkmultable mx* attached to $x \in \mathbf{Z}_K$, return the positive generator of $(x) \cap \mathbf{Z}$.

`GEN zk_scalar_or_multable(GEN nf, GEN x)` given a `t_INT` or `ZC` x, returns a `t_INT` equal to x if the latter is a scalar (`t_INT` or `ZV_isscalar`(x) is 1) and `zk_multable`(nf, x) otherwise. Shallow function.

13.1.11 Number field arithmetic for linear algebra.

The following routines implement multiplication in a commutative R-algebra, generated by $(e_1 = 1, \ldots, e_n)$, and given by a multiplication table M: elements in the algebra are n-dimensional `t_COL`s, and the matrix M is such that for all $1 \leq i, j \leq n$, its column with index $(i-1)n + j$, say (c_k), gives $e_i \cdot e_j = \sum c_k e_k$. It is assumed that e_1 is the neutral element for the multiplication (a convenient optimization, true in practice for all multiplications we needed to implement). If x has any other type than `t_COL` where an algebra element is expected, it is understood as xe_1.

`GEN multable(GEN M, GEN x)` given a column vector x, representing the quantity $\sum_{i=1}^N x_i e_i$, returns the multiplication table by x. Shallow function.

`GEN ei_multable(GEN M, long i)` returns the multiplication table by the i-th basis element e_i. Shallow function.

`GEN tablemul(GEN M, GEN x, GEN y)` returns $x \cdot y$.

`GEN tablesqr(GEN M, GEN x)` returns x^2.

`GEN tablemul_ei(GEN M, GEN x, long i)` returns $x \cdot e_i$.

`GEN tablemul_ei_ej(GEN M, long i, long j)` returns $e_i \cdot e_j$.

`GEN tablemulvec(GEN M, GEN x, GEN v)` given a vector v of elements in the algebra, returns the $x \cdot v[i]$.

The following routines implement naive linear algebra using the *black box field* mechanism:

`GEN nfM_det(GEN nf, GEN M)`

`GEN nfM_inv(GEN nf, GEN M)`

`GEN nfM_ker(GEN nf, GEN M)`

`GEN nfM_mul(GEN nf, GEN A, GEN B)`

`GEN nfM_nfC_mul(GEN nf, GEN A, GEN B)`

13.1.12 Cyclotomic field arithmetic for linear algebra.

The following routines implement modular algorithms in cyclotomic fields. In the prototypes, P is the n-th cyclotomic polynomial Φ_n and M is a `t_MAT` with `t_INT` or `ZX` coefficients, understood modulo P.

`GEN ZabM_ker(GEN M, GEN P, long n)` returns an integral (primitive) basis of the kernel of M.

`GEN ZabM_indexrank(GEN M, GEN P, long n)` return a vector with two `t_VECSMALL` components givin the rank profile of M. Inefficient (but correct) when M does not have almost full column rank.

`GEN ZabM_inv(GEN M, GEN P, long n, GEN *pden)` assume that M is invertible; return N and sets the algebraic integer `*pden` (an integer or a `ZX`, implicitly modulo P) such that $MN =$ `den`$\cdot\mathrm{Id}$.

`GEN ZabM_pseudoinv(GEN M, GEN P, long n, GEN *pv, GEN *pden)` analog of `ZM_pseudoinv`. Not gerepile-safe.

`GEN ZabM_inv_ratlift(GEN M, GEN P, long n, GEN *pden)` return a primitive matrix H such that MH is d times the identity and set `*pden` to d. Uses a multimodular algorithm, attempting rational reconstruction along the way. To be used when you expect that the denominator of M^{-1} is much smaller than $\det M$ else use `ZabM_inv`.

13.1.13 Cyclotomic trace.

Given two positive integers m and n such that $K_m = \mathbf{Q}(\zeta_m) \subset K_n = \mathbf{Q}(\zeta_n)$, these functions implement relative trace computation from K_n to K_m. This is in particular useful for character values.

`GEN Qab_trace_init(long n, long m, GEN Pn, GEN Pm)` assume that `Pn` is `polcyclo`(n), `Pm` is `polcyclo`(m) (both in the same variable), initialize a structure T used in the following routines. Shallow function.

`GEN Qab_tracerel(GEN T, long t, GEN z)` assume T was created by `Qab_trace_init`, t is an integer such that $0 \le t < [K_n : K_m]$ and z belongs to the cyclotomic field $\mathbf{Q}(\zeta_n) = \mathbf{Q}[X]/($`Pn`$)$. Return the normalized relative trace $[K_n : K_m]^{-1}\mathrm{Tr}_{K_n/K_m}(\zeta_n^t z)$. Shallow function.

`GEN QabV_tracerel(GEN T, long t, GEN v)` v being a vector of entries belonging to K_n, apply `Qab_tracerel` to all entries. Shallow function.

`GEN QabM_tracerel(GEN T, long t, GEN m)` m being a matrix of entries belonging to K_n, apply `Qab_tracerel` to all entries. Shallow function.

13.1.14 Elements in factored form.

Computational algebraic theory performs extensively linear algebra on **Z**-modules with a natural multiplicative structure (K^*, fractional ideals in K, $\mathbf{Z}_K^*$, ideal class group), thereby raising elements to horrendously large powers. A seemingly innocuous elementary linear algebra operation like $C_i \leftarrow C_i - 10000C_1$ involves raising entries in C_1 to the 10000-th power. Understandably, it is often more efficient to keep elements in factored form rather than expand every such expression. A *factorization matrix* (or *famat*) is a two column matrix, the first column containing *elements* (arbitrary objects which may be repeated in the column), and the second one contains *exponents* (`t_INT`s, allowed to be 0). By abuse of notation, the empty matrix `cgetg(1, t_MAT)` is recognized as the trivial factorization (no element, no exponent).

Even though we think of a *famat* with columns g and e as one meaningful object when fully expanded as $\prod g[i]^{e[i]}$, *famat*s are basically about concatenating information to keep track of linear algebra: the objects stored in a *famat* need not be operation-compatible, they will not even be compared to each other (with one exception: `famat_reduce`). Multiplying two *famat*s just concatenates their elements and exponents columns. In a context where a *famat* is expected, an object x which is not of type `t_MAT` will be treated as the factorization x^1. The following functions all return *famat*s:

`GEN famat_mul(GEN f, GEN g)` f, g are *famat*, or objects whose type is *not* `t_MAT` (understood as f^1 or g^1). Returns fg. The empty factorization is the neutral element for *famat* multiplication.

`GEN famat_mul_shallow(GEN f, GEN g)` shallow version of `famat_mul`.

`GEN famat_pow(GEN f, GEN n)` n is a `t_INT`. If f is a `t_MAT`, assume it is a *famat* and return f^n (multiplies the exponent column by n). Otherwise, understand it as an element and returns the 1-line *famat* f^n.

`GEN famat_pow_shallow(GEN f, GEN n)` shallow version of `famat_pow`.

`GEN famat_pows_shallow(GEN f, long n)` shallow version of `famat_pow` where n is a small integer.

`GEN famat_mulpow_shallow(GEN f, GEN g, GEN e)` `famat` corresponding to $f \cdot g^e$. Shallow function.

`GEN famat_mulpows_shallow(GEN f, GEN g, long e)` `famat` shallow version of `famat_mulpow` where e is a small integer.

`GEN famat_sqr(GEN f)` returns f^2.

`GEN famat_inv(GEN f)` returns f^{-1}.

`GEN famat_div(GEN f, GEN g)` return f/g.

`GEN famat_inv_shallow(GEN f)` shallow version of `famat_inv`.

`GEN famat_div_shallow(GEN f, GEN g)` return f/g; shallow.

`GEN famat_Z_gcd(GEN M, GEN n)` restrict the *famat* M to the prime power dividing n.

`GEN to_famat(GEN x, GEN k)` given an element x and an exponent k, returns the *famat* x^k.

`GEN to_famat_shallow(GEN x, GEN k)` same, as a shallow function.

`GEN famatV_factorback(GEN v, GEN e)` given a vector of `famat`s v and a `ZV` e return the `famat` $\prod_i v[i]^{e[i]}$. Shallow function.

`GEN famatV_zv_factorback(GEN v, GEN e)` given a vector of `famat`s v and a `zv` e return the `famat` $\prod_i v[i]^{e[i]}$. Shallow function.

`GEN ZM_famat_limit(GEN f, GEN limit)` given a *famat* f with `t_INT` entries, returns a *famat* g with all factors larger than `limit` multiplied out as the last entry (with exponent 1). Shallow function.

Note that it is trivial to break up a *famat* into its two constituent columns: `gel(f,1)` and `gel(f,2)` are the elements and exponents respectively. Conversely, `mkmat2` builds a (shallow) *famat* from two `t_COL`s of the same length.

`GEN famat_reduce(GEN f)` given a *famat* f, returns a *famat* g without repeated elements or 0 exponents, such that the expanded forms of f and g would be equal. Shallow function.

`GEN famat_remove_trivial(GEN f)` given a *famat* f, returns a *famat* g without 0 exponents. Shallow function.

`GEN famatsmall_reduce(GEN f)` as `famat_reduce`, but for exponents given by a `t_VECSMALL`.

`GEN famat_to_nf(GEN nf, GEN f)` You normally never want to do this! This is a simplified form of `nffactorback`, where we do not check the user input for consistency. The elements must be regular algebraic numbers (not *famat*s) over the given number field.

Why should you *not* want to use this function ? You should not need to: most of the functions useful in this context accept *famat*s as inputs, for instance `nfsign`, `nfsign_arch`, `ideallog` and `bnfisunit`. Otherwise, we can hopefully make good use of a quotient operation (modulo a fixed conductor, modulo ℓ-th powers); see the end of Section 13.1.26. If nothing else works, this function is available but is expected to be slow or even overflow the possibilities of the implementation.

`GEN famat_idealfactor(GEN nf, GEN x)` This is a good alternative for `famat_to_nf`, returning the factorization of the ideal generated by x. Since the answer is still given in factorized form, there is no risk of coefficient explosion when the exponents are large. Of course, all components of x must be factored individually.

`GEN famat_nfvalrem(GEN nf, GEN x, GEN pr, GEN *py)` return the valuation v at `pr` of `famat_to_nf`(x), without performing the expansion of course. Notive that the output is a `GEN` since it cannot be assumed to fit into a `long`. If `py` is not `NULL` it contains the `famat` obtained by applying `nfvalrem` to each entry of the first column and copying the second column, with 0 exponents removed. The expanded algebraic number is coprime to `pr` (in fact, all its components are coprime do `pr`) and equal to $x\tau^v$ where τ is the fixed anti-uniformizer for `pr` (`pr_get_tau`).

Caveat. Receiving a *famat* input, `bnfisunit` assumes that it is an actual unit, since this is expensive to check, and normally easy to ensure from the user's side.

13.1.15 Ideal arithmetic.

Conversion to HNF.

`GEN idealhnf(GEN nf, GEN x)` where the argument `nf` is a true *nf* structure. Returns the HNF of the ideal defined by x: x may be an algebraic number (defining a principal ideal), a maximal ideal (as given by `idealprimedec` or `idealfactor`), or a matrix whose columns give generators for the ideal. This last format is complicated, but useful to reduce general modules to the canonical form once in a while:

• if strictly less than $N = [K : Q]$ generators are given, x is the $\mathbf{Z}_K$-module they generate,

• if N or more are given, it is assumed that they form a $\mathbf{Z}$-basis (that the matrix has maximal rank N). This acts as `mathnf` since the $\mathbf{Z}_K$-module structure is (taken for granted hence) not taken into account in this case.

Extended ideals are also accepted, their principal part being discarded.

`GEN idealhnf0(GEN nf, GEN x, GEN y)` returns the HNF of the ideal generated by the two algebraic numbers x and y.

The following low-level functions underlie the above two: they all assume that `nf` is a true *nf* and perform no type checks:

`GEN idealhnf_principal(GEN nf, GEN x)` returns the ideal generated by the algebraic number x.

`GEN idealhnf_shallow(GEN nf, GEN x)` is `idealhnf` except that the result may not be suitable for `gerepile`: if x is already in HNF, we return x, not a copy!

`GEN idealhnf_two(GEN nf, GEN v)` assuming $a = v[1]$ is a nonzero `t_INT` and $b = v[2]$ is an algebraic integer, possibly given in regular representation by a `t_MAT` (the multiplication table by b, see `zk_multable`), returns the HNF of $a\mathbf{Z}_K + b\mathbf{Z}_K$.

Operations.

The basic ideal routines accept all `nf`s (*nf*, *bnf*, *bnr*) and ideals in any form, including extended ideals, and return ideals in HNF, or an extended ideal when that makes sense:

`GEN idealadd(GEN nf, GEN x, GEN y)` returns $x + y$.

`GEN idealdiv(GEN nf, GEN x, GEN y)` returns x/y. Returns an extended ideal if x or y is an extended ideal.

`GEN idealmul(GEN nf, GEN x, GEN y)` returns xy. Returns an extended ideal if x or y is an extended ideal.

`GEN idealsqr(GEN nf, GEN x)` returns x^2. Returns an extended ideal if x is an extended ideal.

`GEN idealinv(GEN nf, GEN x)` returns x^{-1}. Returns an extended ideal if x is an extended ideal.

`GEN idealpow(GEN nf, GEN x, GEN n)` returns x^n. Returns an extended ideal if x is an extended ideal.

`GEN idealpows(GEN nf, GEN ideal, long n)` returns x^n. Returns an extended ideal if x is an extended ideal.

`GEN idealmulred(GEN nf, GEN x, GEN y)` returns an extended ideal equal to xy.

`GEN idealpowred(GEN nf, GEN x, GEN n)` returns an extended ideal equal to x^n.

More specialized routines suffer from various restrictions:

GEN idealdivexact(GEN nf, GEN x, GEN y) returns x/y, assuming that the quotient is an integral ideal. Much faster than idealdiv when the norm of the quotient is small compared to Nx. Strips the principal parts if either x or y is an extended ideal.

GEN idealdivpowprime(GEN nf, GEN x, GEN pr, GEN n) returns $x\mathfrak{p}^{-n}$, assuming x is an ideal in HNF or a rational number, and pr a *prid* attached to $\mathfrak{p}$. Not suitable for gerepileupto since it returns x when $n = 0$. The nf argument must be a true *nf* structure.

GEN idealmulpowprime(GEN nf, GEN x, GEN pr, GEN n) returns $x\mathfrak{p}^{n}$, assuming x is an ideal in HNF or a rational number, and pr a *prid* attached to $\mathfrak{p}$. Not suitable for gerepileupto since it retunrs x when $n = 0$. The nf argument must be a true *nf* structure.

GEN idealprodprime(GEN nf, GEN v) given a list v of prime ideals in *prid* form, return their product. Assume that *nf* is a true *nf* structure.

GEN idealprod(GEN nf, GEN v) given a list v of ideals, return their product.

GEN idealprodval(GEN nf, GEN v, GEN pr) given a list v of ideals return the valuation of their product at the prime ideal pr.

GEN idealHNF_mul(GEN nf, GEN x, GEN y) returns xy, assuming than nf is a true *nf*, x is an integral ideal in HNF and y is an integral ideal in HNF or precompiled form (see below). For maximal speed, the second ideal y may be given in precompiled form $y = [a, b]$, where a is a nonzero t_INT and b is an algebraic integer in regular representation (a t_MAT giving the multiplication table by the fixed element): very useful when many ideals x are going to be multiplied by the same ideal y. This essentially reduces each ideal multiplication to an $N \times N$ matrix multiplication followed by a $N \times 2N$ modular HNF reduction (modulo $xy \cap \mathbf{Z}$).

GEN idealHNF_inv(GEN nf, GEN I) returns I^{-1}, assuming that nf is a true *nf* and x is a fractional ideal in HNF.

GEN idealHNF_inv_Z(GEN nf, GEN I) returns $(I \cap \mathbf{Z}) \cdot I^{-1}$, assuming that nf is a true *nf* and x is an integral fractional ideal in HNF. The result is an integral ideal in HNF.

GEN ideals_by_norm(GEN nf, GEN N) given a true *nf* structure and a integer N, which can also be given by a factorization matrix or (preferably) by a pair $[N, \texttt{factor}(N)]$, return all ideals of norm N in factored form. Not gerepile clean.

Approximation.

GEN idealaddtoone(GEN nf, GEN A, GEN B) given to coprime integer ideals A, B, returns $[a, b]$ with $a \in A$, $b \in B$, such that $a + b = 1$ The result is reduced mod AB, so a, b will be small.

GEN idealaddtoone_i(GEN nf, GEN A, GEN B) as idealaddtoone except that nf must be a true *nf*, and only a is returned.

GEN idealaddtoone_raw(GEN nf, GEN A, GEN B) as idealaddtoone_i except that the reduction mod AB is only performed modulo the lcm of $A \cap \mathbf{Z}$ and $B \cap \mathbf{Z}$, which will increase the size of a.

GEN zkchineseinit(GEN nf, GEN A, GEN B, GEN AB) given two coprime integral ideals A and B (in any form, preferably HNF) and their product AB (in HNF form), initialize a solution to the Chinese remainder problem modulo AB. THe nf argument must be a true *nf* structure.

GEN zkchinese(GEN zkc, GEN x, GEN y) given zkc from zkchineseinit, and x, y two integral elements given as t_INT or ZC, return a z modulo AB such that $z = x \bmod A$ and $z = y \bmod B$.

`GEN zkchinese1(GEN zkc, GEN x)` as `zkchinese` for $y = 1$; useful to lift elements in a nice way from $(\mathbf{Z}_K/A_i)^*$ to $(\mathbf{Z}_K/\prod_i A_i)^*$.

`GEN hnfmerge_get_1(GEN A, GEN B)` given two square upper HNF integral matrices A, B of the same dimension $n > 0$, return a in the image of A such that $1 - a$ is in the image of B. (By abuse of notation we denote 1 the column vector $[1, 0, \ldots, 0]$.) If such an a does not exist, return `NULL`. This is the function underlying `idealaddtoone`.

`GEN idealaddmultoone(GEN nf, GEN v)` given a list of n (globally) coprime integer ideals $(v[i])$ returns an n-dimensional vector a such that $a[i] \in v[i]$ and $\sum a[i] = 1$. If $[K : \mathbf{Q}] = N$, this routine computes the HNF reduction (with $Gl_{nN}(\mathbf{Z})$ base change) of an $N \times nN$ matrix; so it is well worth pruning "useless" ideals from the list (as long as the ideals remain globally coprime).

`GEN idealapprfact(GEN nf, GEN fx)` as `idealappr`, except that x *must* be given in factored form. (This is unchecked.)

`GEN idealcoprime(GEN nf, GEN x, GEN y)`. Given 2 integral ideals x and y, returns an algebraic number α such that αx is an integral ideal coprime to y.

`GEN idealcoprimefact(GEN nf, GEN x, GEN fy)` same as `idealcoprime`, except that y is given in factored form, as from `idealfactor`.

`GEN idealchinese(GEN nf, GEN x, GEN y)`

`GEN idealchineseinit(GEN nf, GEN x)`

13.1.16 Maximal ideals.

The PARI structure attached to maximal ideals is a *prid* (for *pr*ime *id*eal), usually produced by `idealprimedec` and `idealfactor`. In this section, we describe the format; other sections will deal with their daily use.

A *prid* attached to a maximal ideal $\mathfrak{p}$ stores the following data: the underlying rational prime p, the ramification degree $e \geq 1$, the residue field degree $f \geq 1$, a p-uniformizer π with valuation 1 at $\mathfrak{p}$ and valuation 0 at all other primes dividing p and a rescaled "anti-uniformizer" τ used to compute valuations. This τ is an algebraic integer such that τ/p has valuation -1 at $\mathfrak{p}$ and is integral at all other primes; in particular, the valuation of $x \in \mathbf{Z}_K$ is positive if and only if the algebraic integer $x\tau$ is divisible by p (easy to check for elements in `t_COL` form).

`GEN pr_get_p(GEN pr)` returns p. Shallow function.

`GEN pr_get_gen(GEN pr)` returns π. Shallow function.

`long pr_get_e(GEN pr)` returns e.

`long pr_get_f(GEN pr)` returns f.

`GEN pr_get_tau(GEN pr)` returns `zk_scalar_or_multable`(nf, τ), which is the `t_INT` 1 iff p is inert, and a `ZM` otherwise. Shallow function.

`int pr_is_inert(GEN pr)` returns 1 if p is inert, 0 otherwise.

`GEN pr_norm(GEN pr)` returns the norm p^f of the maximal ideal.

`ulong upr_norm(GEN pr)` returns the norm p^f of the maximal ideal, as an `ulong`. Assume that the result does not overflow.

`GEN pr_hnf(GEN pr)` return the HNF of $\mathfrak{p}$.

GEN pr_inv(GEN pr) return the fractional ideal $\mathfrak{p}^{-1}$, in HNF.

GEN pr_inv_p(GEN pr) return the integral ideal $p\mathfrak{p}^{-1}$, in HNF.

GEN idealprimedec(GEN nf, GEN p) list of maximal ideals dividing the prime p.

GEN idealprimedec_limit_f(GEN nf, GEN p, long f) as idealprimedec, limiting the list to primes of residual degree $\leq f$ if f is nonzero.

GEN idealprimedec_limit_norm(GEN nf, GEN p, GEN B) as idealprimedec, limiting the list to primes of norm $\leq B$, which must be a positive t_INT.

GEN idealprimedec_galois(GEN nf, GEN p) return a single prime ideal above p. The nf argument is a true *nf* structure.

GEN idealprimedec_degrees(GEN nf, GEN p) return a (sorted) t_VECSMALL containing the residue degrees $f(\mathfrak{p}/p)$. The nf argument is a true *nf* structure.

GEN idealprimedec_kummer(GEN nf, GEN Ti, long ei, GEN p) let *nf* (true *nf*) correspond to $K = \mathbf{Q}[X]/(T)$ (T monic ZX). Let $T \equiv \prod_i T_i^{e_i} \pmod{p}$ be the factorization of T and let (f, g, h) be as in Dedekind criterion for prime p: $f \equiv \prod T_i$, $g \equiv \prod T_i^{e_i - 1}$, $h = (T - fg)/p$, and let D be the gcd of (f, g, h) in $\mathbf{F}_p[X]$. Let Ti (FpX) be one irreducible factor T_i not dividing D, with ei $= e_i$. This function returns the prime ideal attached to T_i by Kummer / Dedekind criterion, namely $p\mathbf{Z}_K + T_i(\bar{X})\mathbf{Z}_K$, which has ramification index e_i over p. The nf argument is a true *nf* structure. Shallow function.

GEN idealfactor(GEN nf, GEN x) factors the fractional (hence nonzero) ideal x into prime ideal powers; return the factorization matrix.

GEN idealfactor_limit(GEN nf, GEN x, ulong lim) as idealfactor, including only prime ideals above rational primes < lim.

GEN idealfactor_partial(GEN nf, GEN x, GEN L) return partial factorization of fractional ideal x as limited by argument L:

• $L =$ NULL: as idealfactor;

• L a t_INT: as idealfactor_limit;

• L a vector of prime ideals of *nf* and/or rational primes (standing for "all prime ideal divisors of given rational prime") limit factorization to trial division by elements of L; do not include the cofactor.

GEN idealHNF_Z_factor(GEN x, GEN *pvN, GEN *pvZ) given an integral (nonzero) ideal x in HNF, compute both the factorization of Nx and of $x \cap \mathbf{Z}$. This returns the vector of prime divisors of both and sets *pvN and *pvZ to the corresponding t_VECSMALL vector of exponents for the factorization for the Norm and intersection with $\mathbf{Z}$ respetively.

GEN idealHNF_Z_factor_i(GEN x, GEN fa, GEN *pvN, GEN *pvZ) internal variant of idealHNF_Z_factor where fa is either a partial factorization of $x \cap \mathbf{Z}$ $(= x[1,1])$ or NULL. Returns the prime divisors of x above the rational primes in fa and attached vN and vZ. If fa is NULL, use the full factorization, i.e. identical to idealHNF_Z_factor.

GEN nf_pV_to_prV(GEN)nf, GEN P given a vector of rational primes P, return the vector of all prime ideals above the $P[i]$.

`GEN nf_deg1_prime(GEN nf)` let `nf` be a true *nf*. This function returns a degree 1 (unramified) prime ideal not dividing `nf.index`. In fact it returns an ideal above the smallest prime $p \geq [K : \mathbf{Q}]$ satisfying those conditions.

`GEN prV_lcm_capZ(GEN L)` given a vector L of *prid* (maximal ideals) return the squarefree positive integer generating their lcm intersected with $\mathbf{Z}$. Not `gerepile`-safe.

`GEN prV_primes(GEN)`GEN L given a vector of `prid`, return the (sorted) list of rational primes P they divide. Not `gerepile`-clean but suitable for `gerepileupto`.

`GEN pr_uniformizer(GEN pr, GEN F)` given a *prid* attached to $\mathfrak{p}/p$ and F in $\mathbf{Z}$ divisible exactly by p, return an F-uniformizer for `pr`, i.e. a t in $\mathbf{Z}_K$ such that $v_{\mathfrak{p}}(t) = 1$ and $(t, F/\mathfrak{p}) = 1$. Not `gerepile`-safe.

13.1.17 Decomposition groups.

`GEN idealramfrobenius(GEN nf, GEN gal, GEN pr, GEN ram)` Let K be the number field defined by nf and assume $K/\mathbf{Q}$ be a Galois extension with Galois group given `gal=galoisinit(nf)`, and that *pr* is the prime ideal $\mathfrak{P}$ in prid format, and that $\mathfrak{P}$ is ramified, and `ram` is its list of ramification groups as output by `idealramgroups`. This function returns a permutation of `gal.group` which defines an automorphism σ in the decomposition group of $\mathfrak{P}$ such that if p is the unique prime number in $\mathfrak{P}$, then $\sigma(x) \equiv x^p \bmod \mathbf{P}$ for all $x \in \mathbf{Z}_K$.

`GEN idealramfrobenius_aut(GEN nf, GEN gal, GEN pr, GEN ram, GEN aut)` as `idealram-frobenius(nf, gal, pr, ram`.

`GEN idealramgroups_aut(GEN nf, GEN gal, GEN pr, GEN aut)` as `idealramgroups(nf, gal, pr`.

`GEN idealfrobenius_aut(GEN nf, GEN gal, GEN pr, GEN aut)` faster version of `idealfrobe-nius(nf, gal, pr` where `aut` must be equal to `nfgaloispermtobasis(nf, gal)`.

13.1.18 Reducing modulo maximal ideals.

`GEN nfmodprinit(GEN nf, GEN pr)` returns an abstract `modpr` structure, attached to reduction modulo the maximal ideal `pr`, in `idealprimedec` format. From this data we can quickly project any `pr`-integral number field element to the residue field.

`GEN modpr_get_pr(GEN x)` return the `pr` component from a `modpr` structure.

`GEN modpr_get_p(GEN x)` return the p component from a `modpr` structure (underlying rational prime).

`GEN modpr_get_T(GEN x)` return the `T` component from a `modpr` structure: either `NULL` (prime of degree 1) or an irreducible `FpX` defining the residue field over $\mathbf{F}_p$.

In library mode, it is often easier to use directly

`GEN nf_to_Fq_init(GEN nf, GEN *ppr, GEN *pT, GEN *pp)` concrete version of `nfmodprinit`: `nf` and `*ppr` are the inputs, the return value is a `modpr` and `*ppr`, `*pT` and `*pp` are set as side effects.

The input `*ppr` is either a maximal ideal or already a `modpr` (in which case it is replaced by the underlying maximal ideal). The residue field is realized as $\mathbf{F}_p[X]/(T)$ for some monic $T \in \mathbf{F}_p[X]$, and we set `*pT` to T and `*pp` to p. Set $T =$ `NULL` if the prime has degree 1 and the residue field is $\mathbf{F}_p$.

In short, this receives (or initializes) a `modpr` structure, and extracts from it T, p and $\mathfrak{p}$.

`GEN nf_to_Fq(GEN nf, GEN x, GEN modpr)` returns an `Fq` congruent to x modulo the maximal ideal attached to `modpr`. The output is canonical: all elements in a given residue class are represented by the same `Fq`.

`GEN Fq_to_nf(GEN x, GEN modpr)` returns an `nf` element lifting the residue field element x, either a `t_INT` or an algebraic integer in `algtobasis` format.

`GEN modpr_genFq(GEN modpr)` Returns an `nf` element whose image by `nf_to_Fq` is $X \pmod{T}$, if $\deg T > 1$, else 1.

`GEN zkmodprinit(GEN nf, GEN pr)` as `nfmodprinit`, but we assume we will only reduce algebraic integers, hence do not initialize data allowing to remove denominators. More precisely, we can in fact still handle an x whose rational denominator is not 0 in the residue field (i.e. if the valuation of x is nonnegative at all primes dividing p).

`GEN zk_to_Fq_init(GEN nf, GEN *pr, GEN *T, GEN *p)` as `nf_to_Fq_init`, able to reduce only p-integral elements.

`GEN zk_to_Fq(GEN x, GEN modpr)` as `nf_to_Fq`, for a p-integral x.

`GEN nfM_to_FqM(GEN M, GEN nf, GEN modpr)` reduces a matrix of `nf` elements to the residue field; returns an `FqM`.

`GEN FqM_to_nfM(GEN M, GEN modpr)` lifts an `FqM` to a matrix of `nf` elements.

`GEN nfV_to_FqV(GEN A, GEN nf, GEN modpr)` reduces a vector of `nf` elements to the residue field; returns an `FqV` with the same type as `A` (`t_VEC` or `t_COL`).

`GEN FqV_to_nfV(GEN A, GEN modpr)` lifts an `FqV` to a vector of `nf` elements (same type as `A`).

`GEN nfX_to_FqX(GEN Q, GEN nf, GEN modpr)` reduces a polynomial with `nf` coefficients to the residue field; returns an `FqX`.

`GEN FqX_to_nfX(GEN Q, GEN modpr)` lifts an `FqX` to a polynomial with coefficients in `nf`.

The following functions are technical and avoid computing a true `nfmodpr`:

`GEN pr_basis_perm(GEN nf, GEN pr)` given a true *nf* structure and a prime ideal `pr` above p, return as a `t_VECSMALL` the $f(\mathfrak{p}/p)$ indices i such that the `nf.zk`$[i]$ mod p form an $\mathbf{F}_p$-basis of the residue field.

`GEN QXQV_to_FpM(GEN v, GEN T, GEN p)` let p be a positive integer, v be a vector of n polynomials with rational coefficients whose denominators are coprime to p, and T be a `ZX` (preferably monic) of degree d whose leading coefficient is coprime to p. Return the $d \times n$ `FpM` whose columns are the $v[i]$ mod T, p in the canonical basis $1, X, \dots, X^{d-1}$, see `RgX_to_RgC`. This is for instance useful when v contains a $\mathbf{Z}$-basis of the maximal order of a number field $\mathbf{Q}[X]/(P)$, p is a prime not dividing the index of P and T is an irreducible factor of P mod p, attached to a maximal ideal $\mathfrak{p}$: left-multiplication by the matrix maps number field elements (in basis form) to the residue field of $\mathfrak{p}$.

13.1.19 Valuations.

`long nfval(GEN nf, GEN x, GEN P)` return $v_P(x)$

Unsafe functions. assume that P, Q are `prid`.

`long ZC_nfval(GEN x, GEN P)` returns $v_P(x)$, assuming x is a `ZC`, representing a nonzero algebraic integer.

`long ZC_nfvalrem(GEN x, GEN P, GEN *newx)` returns $v = v_P(x)$, assuming x is a `ZC`, representing a nonzero algebraic integer, and sets `*newx` to $x\tau^v$ which is an algebraic integer coprime to p.

`int ZC_prdvd(GEN x, GEN P)` returns 1 is P divides x and 0 otherwise. Assumes that x is a `ZC`, representing an algebraic integer. Faster than computing $v_P(x)$.

`int pr_equal(GEN P, GEN Q)` returns 1 is P and Q represent the same maximal ideal: they must lie above the same p and share the same e, f invariants, but the p-uniformizer and τ element may differ. Returns 0 otherwise.

13.1.20 Signatures.

"Signs" of the real embeddings of number field element are represented in additive notation, using the standard identification $(\mathbf{Z}/2\mathbf{Z}, +) \to (\{-1, 1\}, \times)$, $s \mapsto (-1)^s$.

With respect to a fixed `nf` structure, a selection of real places (a divisor at infinity) is normally given as a `t_VECSMALL` of indices of the roots `nf.roots` of the defining polynomial for the number field. For compatibility reasons, in particular under GP, the (obsolete) `vec01` form is also accepted: a `t_VEC` with `gen_0` or `gen_1` entries.

The following internal functions go back and forth between the two representations for the Archimedean part of divisors (GP: 0/1 vectors, library: list of indices):

`GEN vec01_to_indices(GEN v)` given a `t_VEC` v with `t_INT` entries return as a `t_VECSMALL` the list of indices i such that $v[i] \neq 0$. (Typically used with $0, 1$-vectors but not necessarily so.) If v is already a `t_VECSMALL`, return it: not suitable for `gerepile` in this case.

`GEN vecsmall01_to_indices(GEN v)` as

```
vec01_to_indices(zv_to_ZV(v));
```

`GEN indices_to_vec01(GEN p, long n)` return the 0/1 vector of length n with ones exactly at the positions $p[1], p[2], \ldots$

`GEN nfsign(GEN nf, GEN x)` x being a number field element and `nf` any form of number field, return the $0-1$-vector giving the signs of the r_1 real embeddings of x, as a `t_VECSMALL`. Linear algebra functions like `Flv_add_inplace` then allow keeping track of signs in series of multiplications. The argument `nf` is a true *nf* structure.

If x is a `t_VEC` of number field elements, return the matrix whose columns are the signs of the $x[i]$.

`GEN nfsign_arch(GEN nf, GEN x, GEN arch)` `arch` being a list of distinct real places, either in `vec01` (`t_VEC` with `gen_0` or `gen_1` entries) or `indices` (`t_VECSMALL`) form (see `vec01_to_indices`), returns the signs of x at the corresponding places. This is the low-level function underlying `nfsign`. The argument `nf` is a true *nf* structure.

`int nfchecksigns(GEN nf, GEN x, GEN pl)` pl is a `t_VECSMALL` with r_1 components, all of which are in $\{-1, 0, 1\}$. Return 1 if $\sigma_i(x) pl[i] \geq 0$ for all i, and 0 otherwise.

`GEN nfsign_units(GEN bnf, GEN archp, int add_tu)` `archp` being a divisor at infinity in `indices` form (or `NULL` for the divisor including all real places), return the signs at `archp` of a `bnf.tu` and of system of fundamental units for the field `bnf.fu`, in that order if `add_tu` is set; and in the same order as `bnf.fu` otherwise.

`GEN nfsign_fu(GEN bnf, GEN archp)` returns the signs at `archp` of the fundamental units `bnf.fu`. This is an alias for `nfsign_units` with `add_tu` unset.

`GEN nfsign_tu(GEN bnf, GEN archp)` returns the signs at `archp` of the torsion unit generator `bnf.tu`.

`GEN nfsign_from_logarch(GEN L, GEN invpi, GEN archp)` given L the vector of the $\log \sigma(x)$, where σ runs through the (real or complex) embeddings of some number field, `invpi` being a floating point approximation to $1/\pi$, and `archp` being a divisor at infinity in `indices` form, return the signs of x at the corresponding places. This is the low-level function underlying `nfsign_units`; the latter is actually a trivial wrapper `bnf` structures include the $\log \sigma(x)$ for a system of fundamental units of the field.

`GEN set_sign_mod_divisor(GEN nf, GEN x, GEN y, GEN sarch)` let $f = f_0 f_\infty$ be a divisor, let `sarch` be the output of `nfarchstar(nf, f0, finf)`, let x encode a vector of signs at the places of f_∞ (see below), and let y be a nonzero number field element. Returns z congruent to y mod f_0 (integral if y is) such that z and x have the same signs at f_∞. The argument `nf` is a true *nf* structure.

The following formats are supported for x: a $\{0,1\}$-vector of signs as a `t_VECSMALL` (0 for positive, 1 for negative); `NULL` for a totally positive element (only 0s); a number field element which is replaced by its signature at f_∞.

`GEN nfarchstar(GEN nf, GEN f0, GEN finf)` for a divisor $f = f_0 f_\infty$ represented by the integral ideal `f0` in HNF and the `finf` in `indices` form, returns $(\mathbf{Z}_K/f_\infty)^*$ in a form suitable for computations mod f. See `set_sign_mod_divisor`.

`GEN idealprincipalunits(GEN nf, GEN pr, long e)` returns the multiplicative group $(1 + pr)/(1 + pr^e)$ as an abelian group. Faster than `idealstar` when the norm of pr is large, since it avoids (useless) work in the multiplicative group of the residue field.

13.1.21 Complex embeddings.

`GEN nfembed(GEN nf, GEN x, long k)` returns a floating point approximation of the k-th embedding of x (attached to the k-th complex root in `nf.roots`).

`GEN nf_cxlog(GEN nf, GEN x, long prec)` return the vector of complex logarithmic embeddings $(e_i \mathrm{Log}(\sigma_i X))$ where $e_i = 1$ if $i \le r_1$ and $e_i = 2$ if $r_1 < i \le r_2$ of $X =$ `Q_primpart`(x). Returns `NULL` if loss of accuracy. Not `gerepile`-clean but suitable for `gerepileupto`. Allows x in compact representation, in which case `Q_primpart` is taken componentwise.

`GEN nf_cxlog_normalize(GEN nf, GEN x, long prec)` an *nf* structure attached to a number field K and x from `nf_cxlog`(nf, X) (a column vector of complex logarithmic embeddings with $r_1 + r_2$ components) and let $e = (e_1, \ldots, e_{r_1+r_2})$. Return

$$x - \frac{\log\left(N_{K/\mathbf{Q}} X\right)}{[K:\mathbf{Q}]} e$$

where the imaginary parts are further normalized modulo $2\pi i \cdot e$.

The composition `nf_cxlog` followed by `nf_cxlog_normalize` is a morphism from $(K^*/\mathbf{Q}^*_+, \times)$ to $((\mathbf{C}/2\pi i\mathbf{Z})^{r_1} \times (\mathbf{C}/4\pi i\mathbf{Z})^{r_2}, +)$. Its real part maps the units $\mathbf{Z}_K^*$ to a lattice in the hyperplane $\sum_i x_i = 0$ in $\mathbf{R}^{r_1+r_2}$.

`GEN nfV_cxlog(GEN nf, GEN x, long prec)` applies `nf_cxlog` to each component of the vector x. Returns `NULL` if loss of accuracy for even one component. Not `gerepile`-clean.

`GEN nflogembed(GEN nf, GEN x, GEN *emb, long prec)` return the vector of real logarithmic embeddings $(e_i \mathrm{Log}|\sigma_i x|)$ where $e_i = 1$ if $i \leq r_1$ and $e_i = 2$ if $r_1 < i \leq r_2$. Returns `NULL` if loss of accuracy. Not `gerepile`-clean. If `emb` is non-`NULL` set it to $(e_i \sigma_i x)$. Allows x in compact representation, in which case `emb` is returned in compact representation as well, as a factorization matrix (expanding the factorization may overflowexponents).

13.1.22 Maximal order and discriminant, conversion to `nf` structure.

A number field $K = \mathbf{Q}[X]/(T)$ is defined by a monic $T \in \mathbf{Z}[X]$. The low-level function computing a maximal order is

`void nfmaxord(nfmaxord_t *S, GEN T0, long flag)`, where the polynomial T_0 is squarefree with integer coefficients. Let K be the étale algebra $\mathbf{Q}[X]/(T_0)$ and let $T =$ `ZX_Q_normalize`(T_0), i.e. $T = CT_0(X/L)$ is monic and integral for some $C, Q \in \mathbf{Q}$.

The structure `nfmaxord_t` is initialized by the call; it has the following fields:

```
GEN T0, T, dT, dK; /* T0, T, discriminants of T and K */
GEN unscale; /* the integer L */
GEN index; /* index of power basis in maximal order */
GEN dTP, dTE; /* factorization of |dT|, primes / exponents */
GEN dKP, dKE; /* factorization of |dK|, primes / exponents */
GEN basis; /* Z-basis for maximal order of Q[X]/(T) */
```

The exponent vectors are `t_VECSMALL`. The primes in `dTP` and `dKP` are pseudoprimes, not proven primes. We recommend restricting to $T = T_0$, i.e. either to pass the input polynomial through `ZX_Q_normalize` *before* the call, or to forget about T_0 and go on with the polynomial T; otherwise `unscale` $\neq 1$, all data is expressed in terms of $T \neq T_0$, and needs to be converted to T_0. For instance to convert the basis to $\mathbf{Q}[X]/(T_0)$:

```
RgXV_unscale(S.basis, S.unscale)
```

Instead of passing T (monic `ZX`), one can use the format $[T, \mathit{listP}]$ as in `nfbasis` or `nfinit`, which computes an order which is maximal at a set of primes, but need not be the maximal order.

The `flag` is an or-ed combination of the binary flags, both of them deprecated:

`nf_PARTIALFACT`: do not try to fully factor `dT` and only look for primes less than `primelimit`. In that case, the elements in `dTP` and `dKP` need not all be primes. But the resulting `dK`, `index` and `basis` are correct provided there exists no prime $p >$ `primelimit` such that p^2 divides the field discriminant `dK`. This flag is *deprecated*: the $[T, \mathit{listP}]$ format is safer and more flexible.

`nf_ROUND2`: this flag is *deprecated* and now ignored.

`void nfinit_basic(nfmaxord_t *S, GEN T0)` a wrapper around `nfmaxord` (without the deprecated `flag`) that also accepts number field structures (*nf*, *bnf*, ...) for `T0`.

`GEN nfmaxord_to_nf(nfmaxord_t *S, GEN ro, long prec)` convert an `nfmaxord_t` to an *nf* structure at precision `prec`, where `ro` is `NULL`. The argument `ro` may also be set to a vector with $r_1 + r_2$ components containing the roots of `S->T` suitably ordered, i.e. first r_1 `t_REAL` roots, then r_2 `t_COMPLEX` representing the conguate pairs, but this is *strongly discouraged*: the format is error-prone, and it is hard to compute the roots to the right accuracy in order to achieve `prec` accuracy for the `nf`. This function uses the integer basis `S->basis` as is, *without* performing LLL-reduction. Unless the basis is already known to be reduced, use rather the following higher-level function:

`GEN nfinit_complete(nfmaxord_t *S, long flag, long prec)` convert an `nfmaxord_t` to an *nf* structure at precision `prec`. The `flag` has the same meaning as in `nfinit0`. If `S->basis` is known to be reduced, it will be faster to use `nfmaxord_to_nf`.

`GEN indexpartial(GEN T, GEN dT)` T a monic separable `ZX`, `dT` is either `NULL` (no information) or a multiple of the discriminant of T. Let $K = \mathbf{Q}[X]/(T)$ and $\mathbf{Z}_K$ its maximal order. Returns a multiple of the exponent of the quotient group $\mathbf{Z}_K/(\mathbf{Z}[X]/(T))$. In other word, a *denominator* d such that $dx \in \mathbf{Z}[X]/(T)$ for all $x \in \mathbf{Z}_K$.

`GEN FpX_gcd_check(GEN x, GEN y, GEN D)` let x and y be two coprime polynomials with integer coefficients and let D be a factor of the resultant of x and y; try to factor D by running the Euclidean algorithm on x and y modulo D. This returns `NULL` or a non trivial factor of D. This is the low-level function underlying `poldiscfactors` (applied to x, `ZX_deriv`(x) and the discriminant of x). It succeeds when D has at least two prime divisors p and q such that one sub-resultant of x and y is divisible by p but not by q.

13.1.23 Computing in the class group.

We compute with arbitrary ideal representatives (in any of the various formats seen above), and call

`GEN bnfisprincipal0(GEN bnf, GEN x, long flag)`. The `bnf` structure already contains information about the class group in the form $\oplus_{i=1}^{n}(\mathbf{Z}/d_i\mathbf{Z})g_i$ for canonical integers d_i (with $d_n \mid \ldots \mid d_1$ all > 1) and essentially random generators g_i, which are ideals in HNF. We normally do not need the value of the g_i, only that they are fixed once and for all and that any (nonzero) fractional ideal x can be expressed uniquely as $x = (t)\prod_{i=1}^{n} g_i^{e_i}$, where $0 \le e_i < d_i$, and (t) is some principal ideal. Computing e is straightforward, but t may be very expensive to obtain explicitly. The routine returns (possibly partial) information about the pair $[e, t]$, depending on `flag`, which is an or-ed combination of the following symbolic flags:

• `nf_GEN` tries to compute t. Returns $[e, t]$, with t an empty vector if the computation failed. This flag is normally useless in nontrivial situations since the next two serve analogous purposes in more efficient ways.

• `nf_GENMAT` tries to compute t in factored form, which is much more efficient than `nf_GEN` if the class group is moderately large; imagine a small ideal $x = (t)g^{10000}$: the norm of t has 10000 as many digits as the norm of g; do we want to see it as a vector of huge meaningless integers? The idea is to compute e first, which is easy, then compute (t) as $x\prod g_i^{-e_i}$ using successive `idealmulred`, where the ideal reduction extracts small principal ideals along the way, eventually raised to large powers because of the binary exponentiation technique; the point is to keep this principal part in factored *unexpanded* form. Returns $[e, t]$, with t an empty vector if the computation failed; this should be exceedingly rare, unless the initial accuracy to which `bnf` was computed was ridiculously low (and then `bnfinit` should not have succeeded either). Setting/unsetting `nf_GEN` has no effect when this flag is set.

• `nf_GEN_IF_PRINCIPAL` tries to compute t *only* if the ideal is principal ($e = 0$). Returns `gen_0` if the ideal is not principal. Setting/unsetting `nf_GEN` has no effect when this flag is set, but setting/unsetting `nf_GENMAT` is possible.

• `nf_FORCE` in the above, insist on computing t, even if it requires recomputing a `bnf` from scratch. This is a last resort, and normally the accuracy of a `bnf` can be increased without trouble, but it may be that some algebraic information simply cannot be recovered from what we have: see `bnfnewprec`. It should be very rare, though.

In simple cases where you do not care about t, you may use

`GEN isprincipal(GEN bnf, GEN x)`, which is a shortcut for `bnfisprincipal0(bnf, x, 0)`.

The following low-level functions are often more useful:

`GEN isprincipalfact(GEN bnf, GEN C, GEN L, GEN f, long flag)` is about the same as `bnfisprincipal0` applied to $C\prod L[i]^{f[i]}$, where the $L[i]$ are ideals, the $f[i]$ integers and C is either an ideal or `NULL` (omitted). Make sure to include `nf_GENMAT` in `flag`!

`GEN isprincipalfact_or_fail(GEN bnf, GEN C, GEN L, GEN f)` is for delicate cases, where we must be more clever than `nf_FORCE` (it is used when trying to increase the accuracy of a *bnf*, for instance). If performs

```
isprincipalfact(bnf,C, L, f, nf_GENMAT);
```

but if it fails to compute t, it just returns a `t_INT`, which is the estimated precision (in words, as usual) that would have been sufficient to complete the computation. The point is that `nf_FORCE` does exactly this internally, but goes on increasing the accuracy of the `bnf`, then discarding it, which is a major inefficiency if you intend to compute lots of discrete logs and have selected a precision which is just too low. (It is sometimes not so bad since most of the really expensive data is cached in `bnf` anyway, if all goes well.) With this function, the *caller* may decide to increase the accuracy using `bnfnewprec` (and keep the resulting `bnf`!), or avoid the computation altogether. In any case the decision can be taken at the place where it is most likely to be correct.

`void bnftestprimes(GEN bnf, GEN B)` is an ingredient to certify unconditionnally a `bnf` computed assuming GRH, cf. `bnfcertify`. Running this function successfully proves that the classes of all prime ideals of norm $\leq B$ belong to the subgroup of the class group generated by the factorbase used to compute the `bnf` (equal to the class group under GRH). If the condition is not true, then (GRH is false and) the function will run forever.

If it is known that primes of norm less than B generate the class group (through variants of Minkowski's convex body or Zimmert's twin classes theorems), then the true class group is proven to be a quotient of `bnf.clgp`.

13.1.24 Floating point embeddings, the T_2 quadratic form.

We assume the *nf* is a true `nf` structure, attached to a number field K of degree n and signature (r_1, r_2). We saw that

`GEN nf_get_M(GEN nf)` returns the $(r_1+r_2)\times n$ matrix M giving the embeddings of K, so that if v is an n-th dimensional `t_COL` representing the element $\sum_{i=1}^{n} v[i]w_i$ of K, then `RgM_RgC_mul(M,v)` represents the embeddings of v. Its first r_1 components are real numbers (`t_INT`, `t_FRAC` or `t_REAL`, usually the latter), and the last r_2 are complex numbers (usually of `t_COMPLEX`, but not necessarily for embeddings of rational numbers).

`GEN embed_T2(GEN x, long r1)` assuming x is the vector of floating point embeddings of some algebraic number v, i.e.

```
x = RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v));
```

returns $T_2(v)$. If the floating point embeddings themselves are not needed, but only the values of T_2, it is more efficient to restrict to real arithmetic and use

```
gnorml2( RgM_RgC_mul(nf_get_G(nf), algtobasis(nf,v)));
```

`GEN embednorm_T2(GEN x, long r1)` analogous to `embed_T2`, applied to the `gnorm` of the floating point embeddings. Assuming that

```
x = gnorm( RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v)) );
```

returns $T_2(v)$.

`GEN embed_roots(GEN z, long r1)` given a vector z of r_1+r_2 complex embeddings of the algebraic number v, return the r_1+2r_2 roots of its characteristic polynomial. Shallow function.

`GEN embed_disc(GEN z, long r1, long prec)` given a vector z of r_1+r_2 complex embeddings of the algebraic number v, return a floating point approximation of the discriminant of its characteristic polynomial as a `t_REAL` of precision `prec`.

`GEN embed_norm(GEN x, long r1)` given a vector z of r_1+r_2 complex embeddings of the algebraic number v, return (a floating point approximation of) the norm of v.

13.1.25 Ideal reduction, low level.

In the following routines *nf* is a true `nf`, attached to a number field K of degree n:

`GEN nf_get_Gtwist(GEN nf, GEN v)` assuming v is a `t_VECSMALL` with r_1+r_2 entries, let

$$||x||_v^2 = \sum_{i=1}^{r_1+r_2} 2^{v_i}\varepsilon_i|\sigma_i(x)|^2,$$

where as usual the σ_i are the (real and) complex embeddings and $\varepsilon_i = 1$, resp. 2, for a real, resp. complex place. This is a twisted variant of the T_2 quadratic form, the standard Euclidean form on $K \otimes \mathbf{R}$. In applications, only the relative size of the v_i will matter.

Let $G_v \in M_n(\mathbf{R})$ be a square matrix such that if $x \in K$ is represented by the column vector X in terms of the fixed $\mathbf{Z}$-basis of $\mathbf{Z}_K$ in *nf*, then

$$||x||_v^2 = {}^t(G_vX)\cdot G_vX.$$

(This is a kind of Cholesky decomposition.) This function returns a rescaled copy of G_v, rounded to nearest integers, specifically `RM_round_maxrank`(G_v). Suitable for `gerepileupto`, but does not collect garbage. For convenience, also allow v =`NULL` (`nf_get_roundG`) and v a `t_MAT` as output from the function itself: in both these cases, shallow function.

`GEN nf_get_Gtwist1(GEN nf, long i)`. Simple special case. Returns the twisted G matrix attached to the vector v whose entries are all 0 except the i-th one, which is equal to 10.

`GEN idealpseudomin(GEN x, GEN G)`. Let x, G be two `ZM`s, such that the product Gx is well-defined. This returns a "small" integral linear combinations of the columns of x, given by the LLL-algorithm applied to the lattice Gx. Suitable for `gerepileupto`, but does not collect garbage.

In applications, x is an integral ideal, G approximates a Cholesky form for the T_2 quadratic form as returned by `nf_get_Gtwist`, and we return a small element a in the lattice (x,T_2). This is used to implement `idealred`.

`GEN idealpseudomin_nonscalar(GEN x, GEN G)`. As `idealpseudomin`, but we insist of returning a nonscalar a (`ZV_isscalar` is false), if the dimension of x is > 1.

In the interpretation where x defines an integral ideal on a fixed $\mathbf{Z}_K$ basis whose first element is 1, this means that a is not rational.

`GEN idealpseudominvec(GEN x, GEN G)`. As `idealpseudomin_nonscalar`, but we return about $n^2/2$ nonscalar elements in x with small T_2-norm, where the dimension of x is n.

`GEN idealpseudored(GEN x, GEN G)`. As `idealpseudomin` but we return the full reduced $\mathbf{Z}$-basis of x as a `t_MAT` instead of a single vector.

`GEN idealred_elt(GEN nf, GEN x)` shortcut for

```
idealpseudomin(x, nf_get_roundG(nf))
```

13.1.26 Ideal reduction, high level.

Given an ideal x this means finding a "simpler" ideal in the same ideal class. The public GP function is of course available

`GEN idealred0(GEN nf, GEN x, GEN v)` finds an $a \in K^*$ such that $(a)x$ is integral of small norm and returns it, as an ideal in HNF. What "small" means depends on the parameter v, see the GP description. More precisely, a is returned by `idealpseudomin`$((x_{\mathbf{Z}})x^(-1), G)$ divided by $x_{\mathbf{Z}}$, where $x_{\mathbf{Z}} = (x \cap \mathbf{Z})$ and where G is `nf_get_Gtwist`(nf, v) for $v \neq$ `NULL` and `nf_get_roundG`(nf) otherwise.

Usually one sets $v =$ `NULL` to obtain an element of small T_2 norm in x:

`GEN idealred(GEN nf, GEN x)` is a shortcut for `idealred0(nf,x,NULL)`.

The function `idealred` remains complicated to use: in order not to lose information x must be an extended ideal, otherwise the value of a is lost. There is a subtlety here: the principal ideal (a) is easy to recover, but a itself is an instance of the principal ideal problem which is very difficult given only an *nf* (once a *bnf* structure is available, `bnfisprincipal0` will recover it).

`GEN idealmoddivisor(GEN bnr, GEN x)` A proof-of-concept implementation, useless in practice. If `bnr` is attached to some modulus f, returns a "small" ideal in the same class as x in the ray class group modulo f. The reason why this is useless is that using extended ideals with principal part in a computation, there is a simple way to reduce them: simply reduce the generator of the principal part in $(\mathbf{Z}_K/f)^*$.

`GEN famat_to_nf_moddivisor(GEN nf, GEN g, GEN e, GEN bid)` given a true *nf* attached to a number field K, a *bid* structure attached to a modulus f, and an algebraic number in factored form $\prod g[i]^{e[i]}$, such that $(g[i], f) = 1$ for all i, returns a small element in $\mathbf{Z}_K$ congruent to it mod f. Note that if f contains places at infinity, this includes sign conditions at the specified places.

A simpler case when the conductor has no place at infinity:

`GEN famat_to_nf_modideal_coprime(GEN nf, GEN g, GEN e, GEN f, GEN expo)` as above except that the ideal f is now integral in HNF (no need for a full *bid*), and we pass the exponent of the group $(\mathbf{Z}_K/f)^*$ as `expo`; any multiple will also do, at the expense of efficiency. Of course if a *bid* for f is available, if is easy to extract f and the exact value of `expo` from it (the latter is the first elementary divisor in the group structure). A useful trick: if you set `expo` to *any* positive integer, the result is correct up to `expo`-th powers, hence exact if `expo` is a multiple of the exponent; this is useful when trying to decide whether an element is a square in a residue field for instance! (take `expo`$= 2$).

`GEN nf_to_Fp_coprime(GEN nf, GEN x, GEN modpr)` this low-level function is variant of `famat_to_nf_modideal_coprime`: *nf* is a true *nf* structure, `modpr` is from `zkmodprinit` attached to a prime of degree 1 above the prime number p, and x is either a number field element or a famat factorization matrix. We finally assume that no component of x has a denominator p.

What to do when the $g[i]$ are not coprime to f, but only $\prod g[i]^{e[i]}$ is? Then the situation is more complicated, and we advise to solve it one prime divisor of f at a time. Let v be the valuation attached to a maximal ideal pr:

GEN famat_makecoprime(GEN nf, GEN g, GEN e, GEN pr, GEN prk, GEN expo) returns an element in $(\mathbf{Z}_K/\mathtt{pr}^k)^*$ congruent to the product $\prod g[i]^{e[i]}$, assumed to be globally coprime to pr. As above, expo is any positive multiple of the exponent of $(\mathbf{Z}_K/\mathtt{pr}^k)^*$, for instance $(Nv-1)p^{k-1}$, if p is the underlying rational prime. You may use other values of expo (see the useful trick in famat_to_nf_modideal_coprime).

GEN sunits_makecoprime(GEN g, GEN pr, GEN prk) is a specialized variant that allows to precondition a vector of $g[i]$ assumed to be integral primes or algebraic integers so that it becomes suitable for famat_to_nf_modideal_coprime modulo pr. This is in particular useful for the output of bnf_get_sunits.

GEN Idealstarprk(GEN nf, GEN pr, long k, long flag) same as Idealstar for $I = \mathtt{pr}^k$. The nf argument is a true *nf* structure.

13.1.27 Class field theory.

Under GP, a class-field theoretic description of a number field is given by a triple A, B, C, where the defining set $[A, B, C]$ can have any of the following forms: [*bnr*], [*bnr*, *subgroup*], [*bnf*, *modulus*], [*bnf*, *modulus*, *subgroup*]. You can still use directly all of (libpari's routines implementing) GP's functions as described in Chapter 3, but they are often awkward in the context of libpari programming. In particular, it does not make much sense to always input a triple A, B, C because of the fringe [*bnf*, *modulus*, *subgroup*]. The first routine to call, is thus

GEN Buchray(GEN bnf, GEN mod, long flag) initializes a *bnr* structure from bnf and modulus mod. flag is an or-ed combination of nf_GEN (include generators) and nf_INIT (if omitted, do not return a *bnr*, only the ray class group as an abelian group). In fact, the single most useful value of flag is nf_INIT to initialize a proper *bnr*: omitting nf_GEN saves a lot of time and will not adversely affect any class field theoretic function; adding nf_GEN makes debugging easier. The flag 0 allows to compute only the ray class group structure but will gain little time; if we only need the *order* of the ray class group, then bnrclassno is fastest.

Now we have a proper *bnr* encoding a bnf and a modulus, we no longer need the [*bnf*, *modulus*] and [*bnf*, *modulus*, *subgroup*] forms, which would internally call Buchray anyway. Recall that a subgroup H is given by a matrix in HNF, whose column express generators of H on the fixed generators of the ray class group that stored in our *bnr*. You may also code the trivial subgroup by NULL. It is also allowed to replace H by a character χ of the ray class group modulo mod: it represents the subgroup $\mathrm{Ker}\chi$.

GEN bnr_subgroup_check(GEN bnr, GEN H, GEN *pdeg) given a *bnr* attached to a modulus *mod*, check whether H represents a congruence subgroup (of the ray class group modulo *mod*) and returns a normalized representation: NULL for the trivial subgroup, or in HNF, reduced modulo the elementary divisors of the ray class group. In particular, if H is a character of the ray class group, the returned value is the character kernel. If pdeg is not NULL, *pdeg is set to the degree of the attached class field: the index of H in the ray class group.

void bnr_subgroup_sanitize(GEN *pbnr, GEN *pH) given a *bnr* and a congruence subgroup, make sanity checks and compute the subgroup conductor. Then replace the pair to match the conductor: the *bnr* has the right conductor as modulus, and the subgroup is normalized. Instead of a *bnr*, this function also accepts a *bnf* (gets replaced by the *bnr* with trivial conductor). Instead of a subgroup, the function also accepts an integer N (replaced by $\mathrm{Cl}_f(K)^N$) or a character (replaced by its kernel).

`void bnr_char_sanitize(GEN *pbnr, GEN *pchi)` same idea as `bnr_subgroup_sanitize`: we are given a *bnr* and a ray class character, make sanity checks and update the data to use the conductor as modulus.

`GEN bnrconductor(GEN bnr, GEN H, long flag)` see the documentation of the GP function.

`GEN bnrconductor_factored(GEN bnr, GEN H)` return a pair $[F, fa]$ where F is the conductor and fa is the factorization of the finite part of the conductor. Shallow function.

`GEN bnrconductor_raw(GEN bnr, GEN H)` return the conductor of H. Shallow function.

`long bnrisconductor(GEN bnr, GEN H)` returns 1 is the class field defined by the subgroup H (of the ray class group mod f coded in `bnr`) has conductor f. Returns 0 otherwise.

`GEN ideallog_units(GEN bnf, GEN bid)` return the images of the units generators `bnf.tu` and `bnf.tu` in the finite abelian group $(\mathbf{Z}_K/f)^*$ attached to `bid`.

`GEN ideallog_units0(GEN bnf, GEN bid, GEN N)` let $G = (\mathbf{Z}_K/f)^*$ be the finite abelian group attached to `bid`. Return the images of the units generators `bnf.tu` and `bnf.tu` in G/G^N. If N is `NULL`, same as `ideallog_units`.

`GEN bnrchar_primitive(GEN bnr, GEN chi, GEN bnrc)` Given a normalized character `chi` $=$ $[d, c]$ on `bnr.clgp` (see `char_normalize`) of conductor `bnrc.mod`, compute the primitive character `chic` on `bnrc.clgp` equivalent to `chi`, given as a normalized character $[D, C]$: `chic(bnrc.gen[i])` is $\zeta_D^{C[i]}$, where D is minimal. It is easier to use `bnrconductor_i(bnr,chi,2)`, but the latter recomputes `bnrc` for each new character.

`GEN bnrchar_primitive_raw(GEN bnr, GEN chi, GEN bnrc)` as `bnrchar_primitive`, with `chi` a regular (unnormalized) character on `bnr.clgp` of conductor `bnrc.mod`. Return a regular (unnormalized) primitive character on `bnrc`.

`GEN bnrdisc(GEN bnr, GEN H, long flag)` returns the discriminant and signature of the class field defined by `bnr` and H. See the description of the GP function for details. *flag* is an or-ed combination of the flags `rnf_REL` (output relative data) and `rnf_COND` (return 0 unless the modulus is the conductor).

`GEN ABC_to_bnr(GEN A, GEN B, GEN C, GEN *H, int addgen)` This is a quick conversion function designed to go from the too general (inefficient) A, B, C form to the preferred *bnr*, H form for class fields. Given A, B, C as explained above (omitted entries coded by `NULL`), return the attached *bnr*, and set H to the attached subgroup. If `addgen` is 1, make sure that if the *bnr* needed to be computed, then it contains generators.

13.1.28 Abelian maps. A map $f : A \to B$ between two abelian groups of finite type is given by a triple: $[M, cyc_A, cyc_B]$, where $cyc_A = [a_1, \ldots, a_m]$ and $cyc_B = [b_1, \ldots, b_n]$ are the elementary divisors for A and B (see `ZM_snf`) so that $A = \oplus_{i\leq m}(\mathbf{Z}/a_i\mathbf{Z})g_i$ and $B =\sim \oplus_{j\leq n}(\mathbf{Z}/b_j\mathbf{Z})G_j$. The matrix M gives the image of the generators g_i in terms of the G_j: $(f(g_i))_{i\leq m} = (G_j)_{j\leq n} \cdot M$. The function `bnrmap` returns such a structure.

`GEN bnrsurjection(GEN BNR, GEN bnr)` `BNR` and `bnr` defined over the same field K, for moduli F and f with $f \mid F$, returns the canonical surjection $\mathrm{Cl}_K(F) \to \mathrm{Cl}_K(f)$ as an abelian map. I.e., a triple $[M, cyc_F, cyc_f]$. M gives the image of the fixed ray class group generators of `BNR` in terms of the ones in `bnr`, cyc_F and cyc_f are the cyclic structures of `BNR` and `bnr` respectively (as per `bnr_get_cyc`). Shallow function.

`GEN abmap_kernel(GEN S)` returns the kernel of the abelian map S, ans a matrix H in HNF: the subgroup is $(g_i) \cdot H$.

`GEN abmap_subgroup_image(GEN S, GEN H)` given a subgroup H of A (its generators are the $(g_i)H$); for efficiency, H should be given in canonical form, i.e., as an HNF left divisor of $diag(a_1, \ldots, a_m)$. Returns the subgroup $f(H)$ of B, as an HNF left divisor of $diag(b_1, \ldots, b_n)$.

13.1.29 Grunwald–Wang theorem.

`GEN nfgwkummer(GEN nf, GEN Lpr, GEN Ld, GEN pl, long var)` low-level version of `nfgrunwaldwang`, assuming that `nf` contains suitable roots of unity, and directly using Kummer theory to construct the extension.

`GEN bnfgwgeneric(GEN bnf, GEN Lpr, GEN Ld, GEN pl, long var)` low-level version of `nfgrunwaldwang`, assuming that `bnf` is a `bnfinit` structure, and calling `rnfkummer` to construct the extension.

13.1.30 Relative equations, Galois conjugates.

`GEN nfissquarefree(GEN nf, GEN P)` given P a polynomial with coefficients in *nf*, return 1 is P is squarefree, and 0 otherwise. If is allowed (though less efficient) to replace *nf* by a monic `ZX` defining the field.

`GEN rnfequationall(GEN A, GEN B, long *pk, GEN *pLPRS)` A is either an *nf* type (corresponding to a number field K) or an irreducible `ZX` defining a number field K. B is an irreducible polynomial in $K[X]$. Returns an absolute equation C (over $\mathbf{Q}$) for the number field $K[X]/(B)$. C is the characteristic polynomial of $b + ka$ for some roots a of A and b of B, and k is a small rational integer. Set `*pk` to k.

If `pLPRS` is not `NULL` set it to $[h_0, h_1]$, $h_i \in \mathbf{Q}[X]$, where $h_0 + h_1Y$ is the last nonconstant polynomial in the pseudo-Euclidean remainder sequence attached to $A(Y)$ and $B(X - kY)$, leading to $C = \mathrm{Res}_Y(A(Y), B(X - kY))$. In particular $a := -h_0/h_1$ is a root of A in $\mathbf{Q}[X]/(C)$, and $X - ka$ is a root of B.

`GEN nf_rnfeq(GEN A, GEN B)` wrapper around `rnfequationall` to allow mapping $K \to L$ (`eltup`) and converting elements of L between absolute and relative form (`reltoabs`, `abstorel`), *without* computing a full *rnf* structure, which is useful if the relative integral basis is not required. In fact, since A may be a `t_POL` or an *nf*, the integral basis of the base field is not needed either. The return value is the same as `rnf_get_map`. Shallow function.

`GEN nf_rnfeqsimple(GEN A, GEN B)` as `nf_rnfeq` except some fields are omitted, so that only the `abstorel` operation is supported. Shallow function.

GEN eltabstorel(GEN rnfeq, GEN x) rnfeq is as given by rnf_get_map (but in this case rnfeltabstorel is more robust), nf_rnfeq or nf_rnfeqsimple, return x as an element of L/K, i.e. as a t_POLMOD with t_POLMOD coefficients. Shallow function.

GEN eltabstorel_lift(GEN rnfeq, GEN x) same as eltabstorel, except that x is returned in partially lifted form, i.e. as a t_POL with t_POLMOD coefficients.

GEN eltreltoabs(GEN rnfeq, GEN x) rnfeq is as given by rnf_get_map (but in this case rnfeltreltoabs is more robust) or nf_rnfeq, return x in absolute form.

GEN nf_nfzk(GEN nf, GEN rnfeq) rnfeq as given by nf_rnfeq, nf a true *nf* structure, return a a suitable representation of nf.zk allowing quick computation of the map $K \to L$ by the function nfeltup, *without* computing a full *rnf* structure, which is useful if the relative integral basis is not required. The computed value is the same as in rnf_get_nfzk. Shallow function.

GEN nfeltup(GEN nf, GEN x, GEN zknf) zknf and is initialized by nf_nfzk or rnf_get_nfzk (but in this case rnfeltup is more robust); nf is a true *nf* structure for K, returns $x \in K$ as a (lifted) element of L, in absolute form.

GEN rnfdisc_factored(GEN nf, GEN pol, GEN *pd) variant of rnfdisc returning the relative discriminant ideal *factorization*, and setting *pd to the discriminant as an element in $K^*/(K^*)^2$. Shallow function. The argument nf is a true *nf* structure.

GEN Rg_nffix(const char *f, GEN T, GEN c, int lift) given a ZX T and a “coefficient” c supposedly belonging to $\mathbf{Q}[y]/(T)$, check whether this is a the case and return a cleaned up version of c. The string f is the calling function name, used to report errors.

This means that c must be one of t_INT, t_FRAC, t_POL in the variable y with rational coefficients, or t_POLMOD modulo T which lift to a rational t_POL as above. The cleanup consists in the following improvements:

• t_POL coefficients are reduced modulo T.

• t_POL and t_POLMOD belonging to $\mathbf{Q}$ are converted to rationals, t_INT or t_FRAC.

• if lift is nonzero, convert t_POLMOD to t_POL, and otherwise convert t_POL to t_POLMODs modulo T.

GEN RgX_nffix(const char *f, GEN T, GEN P, int lift) check whether P is a polynomials with coefficients in the number field defined by the absolute equation $T(y) = 0$, where T is a ZX and returns a cleaned up version of P. This checks whether P is indeed a t_POL with variable compatible with coefficients in $\mathbf{Q}[y]/(T)$, i.e.

```
varncmp(varn(P), varn(T)) < 0
```

and applies Rg_nffix to each coefficient.

GEN RgV_nffix(const char *f, GEN T, GEN P, int lift) as RgX_nffix for a vector of coefficients.

GEN polmod_nffix(const char *f, GEN rnf, GEN x, int lift) given a t_POLMOD x supposedly defining an element of *rnf*, check this and perform Rg_nffix cleanups.

GEN polmod_nffix2(const char *f, GEN T, GEN P, GEN x, int lift) as in polmod_nffix, where the relative extension is explicitly defined as $L = (\mathbf{Q}[y]/(T))[x]/(P)$, instead of by an rnf structure.

long numberofconjugates(GEN T, long pinit) returns a quick multiple for the number of **Q**-automorphism of the (integral, monic) t_POL T, from modular factorizations, starting from prime pinit (you can set it to 2). This upper bounds often coincides with the actual number of conjugates. Of course, you should use nfgaloisconj to be sure.

GEN nfroots_if_split(GEN *pt, GEN T) let *pt point either to a number field structure or an irreducible ZX, defining a number field K. Given T a monic squarefree polynomial with coefficients in $\mathbf{Z}_K$, return the list of roots of pol in K if the polynomial splits completely, and NULL otherwise. In other words, this checks whether $K[X]/(T)$ is normal over K (hence Galois since T is separable by assumption).

In the case where *pT is a ZX, the function has to compute internally a conditional nf attached to K , whose nf.zk may not define the maximal order $\mathbf{Z}_K$ (see nfroots); *pT is then replaced by the conditional nf to avoid losing that information.

GEN rnfabelianconjgen(GEN nf, GEN P) *nf* being a number field structure attached to K and P being an irreducible polynomial in $K[X]$. This function returns gen_0 if $L = K[X]/(P)$ is not abelian over K, else it returns a pair (g, o) where g is a vector of K-automorphisms of L generating the abelian group $G = \mathrm{Gal}(L/K)$ and o is a t_VECSMALL of the same length giving the relative orders of the g_i: $o[1]$ is the order of g_1 and for $i \geq 2$, $o[i]$ is the order of g_i in $G/(g_1, \ldots, g_{i-1})$. The length need not be minimal: the $o[i]$ need not be the elementary divisors of G.

13.1.31 Units.

GEN nfrootsof1(GEN nf) returns a two-component vector $[w, z]$ where w is the number of roots of unity in the number field *nf*, and z is a primitive w-th root of unity.

GEN nfcyclotomicunits(GEN nf, GEN zu) where zu is as output by nfrootsof1(nf), return the vector of the cyclotomic units in nf expressed over the integral basis. If $\zeta = \zeta_n$ belongs to the base field (n maximal), this function returns

• (when n is not a prime power) the $\zeta^a - 1$, for all $1 \leq a < n/2$ such that $n/(a, n)$ is not a prime power and a is a strict divisor of n.

• (all n) for p prime, $v_p(n) = k > 0$, the $(z^a - 1)/(z - 1)$, where $z = \zeta^{n/p^k}$, for all $1 < a \leq (p^k - 1)/2$, $(p, a) = 1$.

These are independent modulo torsion if n is a prime power, but not necessarily so otherwise.

GEN sunits_mod_units(GEN bnf, GEN S) return independent generators for $U_S(K)/U(K)$.

13.1.32 Obsolete routines.

Still provided for backward compatibility, but should not be used in new programs. They will eventually disappear.

GEN zidealstar(GEN nf, GEN x) short for Idealstar(nf,x,nf_GEN)

GEN zidealstarinit(GEN nf, GEN x) short for Idealstar(nf,x,nf_INIT)

GEN zidealstarinitgen(GEN nf, GEN x) short for Idealstar(nf,x,nf_GEN|nf_INIT)

GEN idealstar0(GEN nf, GEN x, long flag) short for idealstarmod(nf, ideal, flag, NULL). Use Idealstarmod or Idealstar.

GEN bnrinit0(GEN bnf, GEN ideal, long flag) short for bnrinitmod(bnf,ideal,flag,NULL). Use Buchray or Buchraymod.

GEN buchimag(GEN D, GEN c1, GEN c2, GEN gCO) short for

```
Buchquad(D,gtodouble(c1),gtodouble(c2), /*ignored*/0)
```

GEN buchreal(GEN D, GEN gsens, GEN c1, GEN c2, GEN RELSUP, long prec) short for

```
Buchquad(D,gtodouble(c1),gtodouble(c2), prec)
```

The following use a naming scheme which is error-prone and not easily extensible; besides, they compute generators as per nf_GEN and not nf_GENMAT. Don't use them:

GEN isprincipalforce(GEN bnf, GEN x)

GEN isprincipalgen(GEN bnf, GEN x)

GEN isprincipalgenforce(GEN bnf, GEN x)

GEN isprincipalraygen(GEN bnr, GEN x), use bnrisprincipal.

Variants on polred: use polredbest.

GEN factoredpolred(GEN x, GEN fa)

GEN factoredpolred2(GEN x, GEN fa)

GEN smallpolred(GEN x)

GEN smallpolred2(GEN x), use Polred.

GEN polred0(GEN x, long flag, GEN p)

GEN polredabs(GEN x)

GEN polredabs2(GEN x)

GEN polredabsall(GEN x, long flun)

Superseded by bnrdisclist0:

GEN discrayabslist(GEN bnf, GEN L)

GEN discrayabslistarch(GEN bnf, GEN arch, long bound)

Superseded by idealappr (*flag*is ignored)

GEN idealappr0(GEN nf, GEN x, long flag)

Superseded by bnrconductor_raw or bnrconductormod:

GEN bnrconductor_i(GEN bnr, GEN H, long flag) shallow variant of bnrconductor.

GEN bnrconductorofchar(GEN bnr, GEN chi)

13.2 Galois extensions of Q.

This section describes the data structure output by the function galoisinit. This will be called a gal structure in the following.

13.2.1 Extracting info from a gal structure.

The functions below expect a gal structure and are shallow. See the documentation of galoisinit for the meaning of the member functions.

GEN gal_get_pol(GEN gal) returns gal.pol

GEN gal_get_p(GEN gal) returns gal.p

GEN gal_get_e(GEN gal) returns the integer e such that gal.mod==gal.p^e.

GEN gal_get_mod(GEN gal) returns gal.mod.

GEN gal_get_roots(GEN gal) returns gal.roots.

GEN gal_get_invvdm(GEN gal) gal[4].

GEN gal_get_den(GEN gal) return gal[5].

GEN gal_get_group(GEN gal) returns gal.group.

GEN gal_get_gen(GEN gal) returns gal.gen.

GEN gal_get_orders(GEN gal) returns gal.orders.

13.2.2 Miscellaneous functions.

GEN nfgaloispermtobasis(GEN nf, GEN gal) return the images of the field generator by the automorphisms gal.orders expressed on the integral basis nf.zk.

GEN nfgaloismatrix(GEN nf, GEN s) returns the ZM attached to the automorphism s, seen as a linear operator expressend on the number field integer basis. This allows to use

```
M = nfgaloismatrix(nf, s);
sx = ZM_ZC_mul(M, x);   /* or RgM_RgC_mul(M, x) if x is not integral */
```

instead of

```
sx = nfgaloisapply(nf, s, x);
```

for an algebraic integer x.

GEN nfgaloismatrixapply(GEN nf, GEN M, GEN x) given an automorphism M in nfgaloismatrix form, return the image of x under the automorphism. Variant of galoisapply.

13.3 Quadratic number fields and quadratic forms.

13.3.1 Checks.

`void check_quaddisc(GEN x, long *s, long *mod4, const char *f)` checks whether the `GEN` x is a quadratic discriminant (`t_INT`, not a square, congruent to $0, 1$ modulo 4), and raise an exception otherwise. Set `*s` to the sign of x and `*mod4` to x modulo 4 (0 or 1), unless `mod4` is `NULL`.

`void check_quaddisc_real(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is positive.

`void check_quaddisc_imag(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is negative.

13.3.2 Class number.

Given a D congruent to 0 or 1 modulo 4, let $h(D)$ denote the class number of the order of discriminant D. The function `quadclassunit` uses index calculus and computes $h(D)$ in subexponential time in $\log |D|$ but it assumes the truth of the GRH. For imaginary quadratic orders, it is also comparatively slow for *small* values, say $|D| \leq 10^{18}$. Here are some alternatives:

`GEN classno(GEN D)` corresponds to `qfbclassno(D,0)` and is only useful for $D < 0$, uses a baby-step giant-step technique and runs in time $O(D1/4)$. The result is guaranteed correct for $|D| < 2 \cdot 10^{10}$ and fastest in that range. For larger values of $|D|$, the algorithm is no longer rigorous and may give incorrect results (we know no concrete example); it also becomes relatively less interesting compared to `quadclassunit`.

`GEN classno2(GEN D)` corresponds to `qfbclassno(D,1)` and runs in time $O(D^{1/2})$; the function is provided for testing purposes only since it is never competitive.

`GEN quadclassnoF(GEN D, GEN *pd)` returns $h(D)/h(d)$ where d is the fundamental discriminant attached to D. If `pd` is not `NULL`, set `*pd` to d.

`GEN quadclassno(GEN D)` returns $h(D)$ using Buchmann's algorithm on the order of discriminant D. If D is not fundamental, it will usually be faster to call `coredisc2_fact` and `quadclassnoF_fact` to reduce to this case first.

`long quadclassnos(long D)` returns $h(D)$ using Buchmann's algorithm on the order of discriminant D.

`ulong unegquadclassnoF(ulong x, long *pd)` returns $h(-x)/h(d)$. Set `*pd` to d.

`ulong uposquadclassnoF(ulong x, long *pd)` returns $h(x)/h(d)$. Set `*pd` to d.

`GEN quadclassnoF_fact(GEN D, GEN P, GEN E)` let D be a fundamental discriminant, and $f = \prod_i P[i]^{E[i]}$ be a positive conductor for the order of discriminant Df^2 (P is a `ZV` and E is a `ZV` or `zv`). Returns

$$[O_D^\times : O_{Df^2}^\times] \cdot h(Df^2)/h(d) = f \prod_{p|f} \bigl(1 - (D/p)p^{-1}\bigr).$$

`ulong uquadclassnoF_fact(ulong d, long s, GEN P, GEN E)` let $s = 1$ or -1 be a sign, $D = sd$ be a fundamental discriminant, and $f = \prod_i P[i]^{E[i]}$ be a positive conductor for the order of discriminant Df^2 (P and E are `t_VECSMALL`). Returns

$$[O_D^\times : O_{Df^2}^\times] \cdot h(Df^2)/h(d) = f \prod_{p|f} \bigl(1 - (D/p)p^{-1}\bigr).$$

GEN hclassno(GEN d) returns the Hurwitz-Kronecker class number $H(d)$. These play a central role in trace fomulas and are usually needed for many consecutive values of d. Thus, the function uses a cache so that later calls for *small* consecutive values of d are instantaneous, see getcache. Large values of d ($d > 500000$) call quadclassunit individually and are not memoized.

GEN hclassnoF_fact(GEN P, GEN E, GEN D) return $H(Df^2)/H(D)$ assuming D is a negative fundamental discriminant, where the conductor f is given in factored form: P (ZV) is the list of prime divisors of f and E (t_VECSMALL) their multiplicities.

long uhclassnoF_fact(GEN faf, long D) return $H(Df^2)/H(D)$ assuming D is a negative fundamental discriminant and $d = Df^2$ is an ulong and faf is factoru(d).

GEN hclassno6(GEN d) assuming $d > 0$, returns the integer $6H(d)$. This is a low-level function behind hclassno.

ulong hclassno6u(ulong d) assuming $d > 0$, returns the integer $6H(d)$. Using this function creates (or extends) caches of Hurwitz class numbers and Corediscs of negative integers to speed up consecutive or repeated calls (see getcache). If this is a problem, use:

ulong hclassno6u_no_cache(ulong d) as hclassno6u without creating caches. Existing caches will be used.

13.3.3 t_QFB.

The functions in this section operate on binary quadratic forms of type t_QFB. When specified, a t_QFB argument q attached to an indefinite form can be replaced by the pair $[q, d]$ where the t_REAL d is Shanks's distance.

GEN qfb_1(GEN q) given a t_QFB q, return the unit form q^0.

int qfb_equal1(GEN q) returns 1 if the t_QFB q is the unit form.

13.3.3.1 Reduction.

GEN qfbred(GEN x) reduction of a t_QFB x. Also allow extended indefinite forms.

GEN qfbred_i(GEN x) internal version of qfbred: assume x is a t_QFB.

13.3.3.2 Composition.

GEN qfbcomp(GEN x, GEN y) compose the two t_QFB x and y (with same discriminant), then reduce the result. This is the same as gmul(x,y). Also allow extended indefinite forms.

GEN qfbcomp_i(GEN x, GEN y) internal version of qfbcomp: assume x and y are t_QFB of the same discriminant.

GEN qfbsqr(GEN x) as qfbcomp(x,x).

GEN qfbsqr_i(GEN x) as qfbcomp_i(x,y).

Same as above, *without* reducing the result:

GEN qfbcompraw(GEN x, GEN y) compose two t_QFBs, without reducing the result. Also allow extended indefinite forms.

GEN qfbcompraw_i(GEN x, GEN y) internal version of qfbcompraw: assume x and y are t_QFB of the same discriminant.

13.3.3.3 Powering.

`GEN qfbpow(GEN x, GEN n)` computes x^n and reduce the result. Also allow extended indefinite forms.

`GEN qfbpows(GEN x, long n)` computes x^n and reduce the result. Also allow extended indefinite forms.

`GEN qfbpow_i(GEN x, GEN n)` internal version of `qfbcomp`. Assume x is a `QFB`.

`GEN qfbpowraw(GEN x, long n)` compute x^n (pure composition, no reduction), for a `t_QFB` x. Also allow indefinite forms.

13.3.3.4 Order, discrete log.

`GEN qfi_order(GEN q, GEN o)` assuming that the imaginary `t_QFB` q has order dividing o, compute its order in the class group. The order can be given in all formats allowed by generic discrete log functions, the preferred format being `[ord, fa]` (`t_INT` and its factorization).

`GEN qfi_log(GEN a, GEN g, GEN o)` given an imaginary `t_QFB` a and assuming that the `t_QFB` g has order o, compute an integer k such that $a^k = g$. Return `cgetg(1, t_VEC)` if there are no solutions. Uses a generic Pollig-Hellman algorithm, then either Shanks (small o) or Pollard rho (large o) method. The order can be given in all formats allowed by generic discrete log functions, the preferred format being `[ord, fa]` (`t_INT` and its factorization).

`GEN qfi_Shanks(GEN a, GEN g, long n)` given an imaginary `t_QFB` a and assuming that the `t_QFB` g has (small) order n, compute an integer k such that $a^k = g$. Return `cgetg(1, t_VEC)` if there are no solutions. Directly uses Shanks algorithm, which is inefficient when n is composite.

13.3.3.5 Solve, Cornacchia.

The following functions underly `qfbsolve`; p denotes a prime number.

`GEN qfisolvep(GEN Q, GEN p)` solves $Q(x,y) = p$ over the integers, for an imaginary `t_QFB` Q. Return `gen_0` if there are no solutions.

`GEN qfrsolvep(GEN Q, GEN p)` solves $Q(x,y) = p$ over the integers, for a real `t_QFB` Q. Return `gen_0` if there are no solutions.

`long cornacchia(GEN d, GEN p, GEN *px, GEN *py)` solves $x^2 + dy^2 = p$ over the integers, where $d > 0$ is congruent to 0 or 3 modulo 4. Return 1 if there is a solution (and store it in `*x` and `*y`), 0 otherwise.

`long cornacchia2(GEN d, GEN p, GEN *px, GEN *py)` as `cornacchia`, for the equation $x^2+dy^2 = 4p$.

`long cornacchia2_sqrt(GEN d, GEN p, GEN b, GEN *px, GEN *py)` as `cornacchia2`, where $p > 2$ and b is the smallest squareroot of d modulo p.

13.3.3.6 Prime forms.

`GEN primeform_u(GEN D, ulong p)` `t_QFB` of discriminant D whose first coefficient is the prime p, assuming $(D/p) \geq 0$.

`GEN primeform(GEN D, GEN p)`

13.3.4 Efficient real quadratic forms. Unfortunately, real `t_QFB`s are very inefficient, and are only provided for backward compatibility.

• they do not contain needed quantities, which are thus constantly recomputed (the discriminant square root $\sqrt{D}$ and its integer part),

• the distance component is stored in logarithmic form, which involves computing one extra logarithm per operation. It is much more efficient to store its exponential, computed from ordinary multiplications and divisions (taking exponent overflow into account), and compute its logarithm at the very end.

Internally, we have two representations for real quadratic forms:

• `qfr3`, a container $[a, b, c]$ with at least 3 entries: the three coefficients; the idea is to ignore the distance component.

• `qfr5`, a container with at least 5 entries $[a, b, c, e, d]$: the three coefficients a `t_REAL` d and a `t_INT` e coding the distance component $2^{Ne}d$, in exponential form, for some large fixed N.

It is a feature that `qfr3` and `qfr5` have no specified length or type. It implies that a `qfr5` or `t_QFB` will do whenever a `qfr3` is expected. Routines using these objects require a global context, provided by a `struct qfr_data *`:

```
struct qfr_data {
  GEN D;        /* discriminant, t_INT   */
  GEN sqrtD;    /* sqrt(D), t_REAL       */
  GEN isqrtD;   /* floor(sqrt(D)), t_INT */
};
```

`void qfr_data_init(GEN D, long prec, struct qfr_data *S)` given a discriminant $D > 0$, initialize S for computations at precision `prec` ($\sqrt{D}$ is computed to that initial accuracy).

All functions below are shallow, and not stack clean.

`GEN qfr3_comp(GEN x, GEN y, struct qfr_data *S)` compose two `qfr3`, reducing the result.

`GEN qfr3_compraw(GEN x, GEN y)` as `qfr3_comp`, without reducing the result.

`GEN qfr3_pow(GEN x, GEN n, struct qfr_data *S)` compute x^n, reducing along the way.

`GEN qfr3_red(GEN x, struct qfr_data *S)` reduce x.

`GEN qfr3_rho(GEN x, struct qfr_data *S)` perform one reduction step; `qfr3_red` just performs reduction steps until we hit a reduced form.

`GEN qfr3_to_qfr(GEN x, GEN d)` recover an ordinary `t_QFB` from the `qfr3` x, adding disriminant component d.

Before we explain `qfr5`, recall that it corresponds to an ideal, that reduction corresponds to multiplying by a principal ideal, and that the distance component is a clever way to keep track of these principal ideals. More precisely, reduction consists in a number of reduction steps, going from the form (a, b, c) to $\rho(a, b, c) = (c, -b \bmod 2c, *)$; the distance component is multiplied by (a floating point approximation to) $(b + \sqrt{D})/(b - \sqrt{D})$.

`GEN qfr5_comp(GEN x, GEN y, struct qfr_data *S)` compose two `qfr5`, reducing the result, and updating the distance component.

`GEN qfr5_compraw(GEN x, GEN y)` as `qfr5_comp`, without reducing the result.

GEN qfr5_pow(GEN x, GEN n, struct qfr_data *S) compute x^n, reducing along the way.

GEN qfr5_red(GEN x, struct qfr_data *S) reduce x.

GEN qfr5_rho(GEN x, struct qfr_data *S) perform one reduction step.

GEN qfr5_dist(GEN e, GEN d, long prec) decode the distance component from exponential (qfr5-specific) to logarithmic form (true Shanks's distance).

GEN qfr_to_qfr5(GEN x, long prec) convert a real t_QFB to a qfr5 with initial trivial distance component (= 1).

GEN qfr5_to_qfr(GEN x, GEN d), assume x is a qfr5 and d is NULL or the original distance component of some real t_QFB. Convert x to a t_QFB, with the correct (logarithmic) distance component if d is not NULL.

13.4 Linear algebra over Z.

13.4.1 Hermite and Smith Normal Forms.

GEN ZM_hnf(GEN x) returns the upper triangular Hermite Normal Form of the ZM x (removing 0 columns), using the ZM_hnfall algorithm. If you want the true HNF, use ZM_hnfall(x, NULL, 0).

GEN ZM_hnfmod(GEN x, GEN d) returns the HNF of the ZM x (removing 0 columns), assuming the t_INT d is a multiple of the determinant of x. This is usually faster than ZM_hnf (and uses less memory) if the dimension is large, > 50 say.

GEN ZM_hnfmodid(GEN x, GEN d) returns the HNF of the ZM x concatenated with the diagonal matrix with diagonal d, where d is a vector of integers of compatible dimension. Variant: if d is a t_INT, then concatenate $d\mathrm{Id}$.

GEN ZM_hnfmodprime(GEN x, GEN p) returns the HNF of the matrix $(x \mid p\mathrm{Id})$ (removing 0 columns), for a ZM x and a prime number p. The algorithm involves only $\mathbf{F}_p$-linear algebra and is is faster than ZM_hnfmodid (which will call it when d is prime).

GEN ZM_hnfmodall(GEN x, GEN d, long flag) low-level function underlying the ZM_hnfmod variants. If flag is 0, calls ZM_hnfmod(x,d); flag is an or-ed combination of:

• hnf_MODID call ZM_hnfmodid instead of ZM_hnfmod,

• hnf_PART return as soon as we obtain an upper triangular matrix, saving time. The pivots are nonnegative and give the diagonal of the true HNF, but the entries to the right of the pivots need not be reduced, i.e. they may be large or negative.

• hnf_CENTER returns the centered HNF, where the entries to the right of a pivot p are centered residues in $[-p/2, p/2[$, hence smallest possible in absolute value, but possibly negative.

GEN ZM_hnfmodall_i(GEN x, GEN d, long flag) as ZM_hnfmodall without final garbage collection. Not gerepile-safe.

GEN ZM_hnfall(GEN x, GEN *U, long remove) returns the upper triangular HNF H of the ZM x; if U is not NULL, set if to the matrix U such that $xU = H$. If remove $= 0$, H is the true HNF, including 0 columns; if remove $= 1$, delete the 0 columns from H but do not update U accordingly (so that the integer kernel may still be recovered): we no longer have $xU = H$; if remove $= 2$, remove 0 columns from H and update U so that $xU = H$. The matrix U is square and invertible unless `remove` = 2.

This routine uses a naive algorithm which is potentially exponential in the dimension (due to coefficient explosion) but is fast in practice, although it may require lots of memory. The base change matrix U may be very large, when the kernel is large.

`GEN ZM_hnfall_i(GEN x, GEN *U, long remove)` as `ZM_hnfall` without final garbage collection. Not `gerepile`-safe.

`GEN ZM_hnfperm(GEN A, GEN *ptU, GEN *ptperm)` returns the hnf $H = PAU$ of the matrix PA, where P is a suitable permutation matrix, and $U \in \mathrm{Gl}_n(\mathbf{Z})$. P is chosen so as to (heuristically) minimize the size of U; in this respect it is less efficient than `ZM_hnflll` but usually faster. Set `*ptU` to U and `*pterm` to a `t_VECSMALL` representing the row permutation attached to $P = (\delta_{i,\texttt{perm}[i]}$. If `ptU` is set to `NULL`, U is not computed, saving some time; although useless, setting `ptperm` to `NULL` is also allowed.

`GEN ZM_hnf_knapsack(GEN x)` given a `ZM` x, compute its HNF h. Return h if it has the knapsack property: every column contains only zeroes and ones and each row contains a single 1; return `NULL` otherwise. Not suitable for gerepile.

`GEN ZM_hnflll(GEN x, GEN *U, int remove)` returns the HNF H of the `ZM` x; if U is not `NULL`, set if to the matrix U such that $xU = H$. The meaning of `remove` is the same as in `ZM_hnfall`.

This routine uses the LLL variant of Havas, Majewski and Mathews, which is polynomial time, but rather slow in practice because it uses an exact LLL over the integers instead of a floating point variant; it uses polynomial space but lots of memory is needed for large dimensions, say larger than 300. On the other hand, the base change matrix U is essentially optimally small with respect to the L_2 norm.

`GEN ZM_hnfcenter(GEN M)`. Given a `ZM` in HNF M, update it in place so that nondiagonal entries belong to a system of *centered* residues. Not suitable for gerepile.

Some direct applications: the following routines apply to upper triangular integral matrices; in practice, these come from HNF algorithms.

`GEN hnf_divscale(GEN A, GEN B, GEN t)` A an upper triangular `ZM`, B a `ZM`, t an integer, such that $C := tA^{-1}B$ is integral. Return C.

`GEN hnf_invscale(GEN A, GEN t)` A an upper triangular `ZM`, t an integer such that $C := tA^{-1}$ is integral. Return C. Special case of `hnf_divscale` when B is the identity matrix.

`GEN hnf_solve(GEN A,  GEN B)` A a `ZM` in upper HNF (not necessarily square), B a `ZM` or `ZC`. Return $A^{-1}B$ if it is integral, and `NULL` if it is not.

`GEN hnf_invimage(GEN A, GEN b)` A a `ZM` in upper HNF (not necessarily square), b a `ZC`. Return $A^{-1}B$ if it is integral, and `NULL` if it is not.

`int hnfdivide(GEN A, GEN B)` A and B are two upper triangular `ZM`. Return 1 if $A^{-1}B$ is integral, and 0 otherwise.

Smith Normal Form.

`GEN ZM_snf(GEN x)` returns the Smith Normal Form (vector of elementary divisors) of the `ZM` x.

`GEN ZM_snfall(GEN x, GEN *U, GEN *V)` returns `ZM_snf(x)` and sets U and V to unimodular matrices such that $U\,x\,V = D$ (diagonal matrix of elementary divisors). Either (or both) U or V may be `NULL` in which case the corresponding matrix is not computed.

`GEN ZV_snfall(GEN d, GEN *U, GEN *V)` here d is a `ZV`; same as `ZM_snfall` applied to `diagonal(d)`, but faster.

`GEN ZM_snfall_i(GEN x, GEN *U, GEN *V, long flag)` low level version of `ZM_snfall`:

• if the first bit of *flag* is 0, return a diagonal matrix (as in `ZM_snfall`), else a vector of elementary divisors (as in `ZM_snf`).

• if the second bit of *flag* is 1, assume that x is invertible and allow U and V to have determinant congruent to 1 modulo d, where d is the largest elementary divisor of x. Rationale: the finite group $G = \mathbf{Z}^n/\Im x$ has exponent d and we are only interested in the action of U, V as they act on G not in genuine unimodular matries. (See `ZM_snf_group`.)

`void ZM_snfclean(GEN d, GEN U, GEN V)` assuming d, U, V come from `d = ZM_snfall(x, &U, &V)`, where U or V may be `NULL`, cleans up the output in place. This means that elementary divisors equal to 1 are deleted and U, V are updated. This also works when d is a `t_VEC` of elementary divisors. The output is not suitable for `gerepileupto`.

`void ZV_snfclean(GEN d)` assuming d is a `t_VEC` of elementary divisors, return a shortened version where divisors equal to 1 are deleted. The output is not suitable for `gerepileupto`; we return d itself if no divisor is 1.

`void ZV_snf_trunc(GEN D)` given a vector D of elementary divisors (i.e. a `ZV` such that $d_i \mid d_{i+1}$), truncate it *in place* to leave out the trivial divisors (equal to 1).

`GEN ZM_snf_group(GEN H, GEN *U, GEN *Uinv)` this function computes data to go back and forth between an abelian group (of finite type) given by generators and relations, and its canonical SNF form. Given an abstract abelian group with generators $g = (g_1, \dots, g_n)$ and a vector $X = (x_i) \in \mathbf{Z}^n$, we write gX for the group element $\sum_i x_i g_i$; analogously if M is an $n \times r$ integer matrix gM is a vector containing r group elements. The group neutral element is 0; by abuse of notation, we still write 0 for a vector of group elements all equal to the neutral element. The input is a full relation matrix H among the generators, i.e. a `ZM` (not necessarily square) such that $gX = 0$ for some $X \in \mathbf{Z}^n$ if and only if X is in the integer image of H, so that the abelian group is isomorphic to $\mathbf{Z}^n/\mathrm{Im} H$. *The routine assumes that H is in HNF;* replace it by its HNF if it is not the case. (Of course this defines the same group.)

Let G a minimal system of generators in SNF for our abstract group: if the d_i are the elementary divisors $(\dots \mid d_2 \mid d_1)$, each G_i has either infinite order ($d_i = 0$) or order $d_i > 1$. Let D the matrix with diagonal (d_i), then

$$GD = 0, \quad G = gU_{\mathrm{inv}}, \quad g = GU,$$

for some integer matrices U and U_{inv}. Note that these are not even square in general; even if square, there is no guarantee that these are unimodular: they are chosen to have minimal entries given the known relations in the group and only satisfy $D \mid (UU_{\mathrm{inv}} - \mathrm{Id})$ and $H \mid (U_{\mathrm{inv}}U - \mathrm{Id})$.

The function returns the vector of elementary divisors (d_i); if `U` is not `NULL`, it is set to U; if `Uinv` is not `NULL` it is set to U_{inv}. The function is not memory clean.

`GEN ZV_snf_group(GEN d, GEN *newU, GEN *newUi)`, here d is a `ZV`; same as `ZM_snf_group` applied to `diagonal(d)`, but faster.

`GEN ZV_snf_gcd(GEN v, GEN N)` given a vector v of integers and a positive integer N, return the vector whose entries are the gcds $(v[i], N)$. Use case: if v gives the cyclic components for some abelian group G of finite type, then this returns the structure of the finite groupe G/G^N.

The following functions compute the p^n-rank of abelian groups given a vector of elementary divisors and underly `snfrank`:

`long ZV_snf_rank(GEN D, GEN p)` assume D is a `ZV` and p is a `t_INT`.

`long ZV_snf_rank_u(GEN D, ulong p)` assume D is a `ZV`.

`long zv_snf_rank(GEN D, ulong p)` assume D is a `zv`.

The following routines underly the various `matrixqz` variants. In all case the $m \times n$ `t_MAT` x is assumed to have rational (`t_INT` and `t_FRAC`) coefficients

`GEN QM_ImQ(GEN x)` returns a basis for $\mathrm{Im}_{\mathbf{Q}} x \cap \mathbf{Z}^n$.

`GEN QM_ImZ(GEN x)` returns a basis for $\mathrm{Im}_{\mathbf{Z}} x \cap \mathbf{Z}^n$.

`GEN QM_ImQ_hnf(GEN x)` returns an HNF basis for $\mathrm{Im}_{\mathbf{Q}} x \cap \mathbf{Z}^n$.

`GEN QM_ImZ_hnf(GEN x)` returns an HNF basis for $\mathrm{Im}_{\mathbf{Z}} x \cap \mathbf{Z}^n$.

`GEN QM_ImQ_hnfall(GEN A, GEN *pB, long remove)` as `QM_ImQ_hnf`, further returning the transformation matrix as in `ZM_hnfall`.

`GEN QM_ImZ_hnfall(GEN A, GEN *pB, long remove)` as `QM_ImZ_hnf`, further returning the transformation matrix as in `ZM_hnfall`.

`GEN QM_ImQ_all(GEN A, GEN *pB, long remove, long hnf)` as `QM_ImQ`, further returning the transformation matrix as in `ZM_hnfall`, and returning an HNF basis if `hnf` is nonzero.

`GEN QM_ImZ_all(GEN A, GEN *pB, long remove, long hnf)` as `QM_ImZ`, further returning the transformation matrix as in `ZM_hnfall`, and returning an HNF basis if `hnf` is nonzero.

`GEN QM_minors_coprime(GEN x, GEN D)`, assumes $m \geq n$, and returns a matrix in $M_{m,n}(\mathbf{Z})$ with the same $\mathbf{Q}$-image as x, such that the GCD of all $n \times n$ minors is coprime to D; if D is `NULL`, we want the GCD to be 1.

The following routines are simple wrappers around the above ones and are normally useless in library mode:

`GEN hnf(GEN x)` checks whether x is a `ZM`, then calls `ZM_hnf`. Normally useless in library mode.

`GEN hnfmod(GEN x, GEN d)` checks whether x is a `ZM`, then calls `ZM_hnfmod`. Normally useless in library mode.

`GEN hnfmodid(GEN x, GEN d)` checks whether x is a `ZM`, then calls `ZM_hnfmodid`. Normally useless in library mode.

`GEN hnfall(GEN x)` calls `ZM_hnfall(x, &U, 1)` and returns $[H, U]$. Normally useless in library mode.

`GEN hnflll(GEN x)` calls `ZM_hnflll(x, &U, 1)` and returns $[H, U]$. Normally useless in library mode.

`GEN hnfperm(GEN x)` calls `ZM_hnfperm(x, &U, &P)` and returns $[H, U, P]$. Normally useless in library mode.

`GEN smith(GEN x)` checks whether x is a `ZM`, then calls `ZM_snf`. Normally useless in library mode.

`GEN smithall(GEN x)` checks whether x is a `ZM`, then calls `ZM_snfall(x, &U, &V)` and returns $[U, V, D]$. Normally useless in library mode.

Some related functions over $K[X]$, K a field:

`GEN gsmith(GEN A)` the input matrix must be square, returns the elementary divisors.

`GEN gsmithall(GEN A)` the input matrix must be square, returns the $[U, V, D]$, D diagonal, such that $UAV = D$.

`GEN RgM_hnfall(GEN A, GEN *pB, long remove)` analogous to `ZM_hnfall`.

`GEN smithclean(GEN z)` cleanup the output of `smithall` or `gsmithall` (delete elementary divisors equal to 1, updating base change matrices).

13.4.2 The LLL algorithm.

The basic GP functions and their immediate variants are normally not very useful in library mode. We briefly list them here for completeness, see the documentation of `qflll` and `qflllgram` for details:

- `GEN qflll0(GEN x, long flag)`

`GEN lll(GEN x)` *flag*$= 0$

`GEN lllint(GEN x)` *flag*$= 1$

`GEN lllkerim(GEN x)` *flag*$= 4$

`GEN lllkerimgen(GEN x)` *flag*$= 5$

`GEN lllgen(GEN x)` *flag*$= 8$

- `GEN qflllgram0(GEN x, long flag)`

`GEN lllgram(GEN x)` *flag*$= 0$

`GEN lllgramint(GEN x)` *flag*$= 1$

`GEN lllgramkerim(GEN x)` *flag*$= 4$

`GEN lllgramkerimgen(GEN x)` *flag*$= 5$

`GEN lllgramgen(GEN x)` *flag*$= 8$

The basic workhorse underlying all integral and floating point LLLs is

`GEN ZM_lll(GEN x, double D, long flag)`, where x is a `ZM`; $D \in]1/4, 1[$ is the Lovász constant determining the frequency of swaps during the algorithm: a larger values means better guarantees for the basis (in principle smaller basis vectors) but longer running times (suggested value: $D = 0.99$).

Important. This function does not collect garbage and its output is not suitable for either `gerepile` or `gerepileupto`. We expect the caller to do something simple with the output (e.g. matrix multiplication), then collect garbage immediately.

`flag` is an or-ed combination of the following flags:

- `LLL_GRAM`. If set, the input matrix x is the Gram matrix ${}^t vv$ of some lattice vectors v.

- `LLL_INPLACE`. Incompatible with `LLL_GRAM`. If unset, we return the base change matrix U, otherwise the transformed matrix xU. Implies `LLL_IM` (see below).

- `LLL_KEEP_FIRST`. The first vector in the output basis is the same one as was originally input. Provided this is a shortest nonzero vector of the lattice, the output basis is still LLL-reduced. This is used to reduce maximal orders of number fields with respect to the T_2 quadratic form, to ensure that the first vector in the output basis corresponds to 1 (which is a shortest vector).

- `LLL_COMPATIBLE`. DEPRECATED. This is now a no-op.

The last three flags are mutually exclusive, either 0 or a single one must be set:

- `LLL_KER` If set, only return a kernel basis K (not LLL-reduced).

- `LLL_IM` If set, only return an LLL-reduced lattice basis T. (This is implied by `LLL_INPLACE`).

- `LLL_ALL` If set, returns a 2-component vector $[K, T]$ corresponding to both kernel and image.

`GEN lllfp(GEN x, double D, long flag)` is a variant for matrices with inexact entries: x is a matrix with real coefficients (types `t_INT`, `t_FRAC` and `t_REAL`), D and *flag* are as in `ZM_lll`. The matrix is rescaled, rounded to nearest integers, then fed to `ZM_lll`. The flag `LLL_INPLACE` is still accepted but less useful (it returns an LLL-reduced basis attached to rounded input, instead of an exact base change matrix).

`GEN ZM_lll_norms(GEN x, double D, long flag, GEN *ptB)` slightly more general version of `ZM_lll`, setting `*ptB` to a vector containing the squared norms of the Gram-Schmidt vectors (b_i^*) attached to the output basis (b_i), $b_i^* = b_i + \sum_{j<i} \mu_{i,j} b_j^*$.

`GEN lllintpartial_inplace(GEN x)` given a `ZM` x of maximal rank, returns a partially reduced basis (b_i) for the space spanned by the columns of x: $|b_i \pm b_j| \geq |b_i|$ for any two distinct basis vectors b_i, b_j. This is faster than the LLL algorithm, but produces much larger bases.

`GEN lllintpartial(GEN x)` as `lllintpartial_inplace`, but returns the base change matrix U from the canonical basis to the b_i, i.e. xU is the output of `lllintpartial_inplace`.

`GEN RM_round_maxrank(GEN G)` given a matrix G with real floating point entries and independent columns, let G_e be the rescaled matrix $2^e G$ rounded to nearest integers, for $e \geq 0$. Finds a small e such that the rank of G_e is equal to the rank of G (its number of columns) and return G_e. This is useful as a preconditioning step to speed up LLL reductions, see `nf_get_Gtwist`. Suitable for `gerepileupto`, but does not collect garbage.

`GEN Hermite_bound(long n, long prec)` return a majoration of γ_n^n where γ_n is the Hermite constant for lattices of dimension n. The bound is sharp in dimension $n \leq 8$ and $n = 24$.

13.4.3 Linear dependencies.

The following functions underly the `lindep` GP function:

`GEN lindep(GEN v)` real/complex entries, guess that about only the 80% leading bits of the input are correct.

`GEN lindep_bit(GEN v, long b)` real/complex entries, explicit form of the above: multiply the input by 2^b and round to nearest integer before looking for a linear dependency. Truncating dubious bits allows to find better relations.

`GEN lindepfull_bit(GEN v, long b)` as `lindep_bit` but return a matrix M with $n = \#v$ columns and r rows, with $r = n+1$ (if v is real) or $n+2$ (general case) which is an LLL-reduced basis of the lattice formed by concatenating vertically an identity matrix and the floor of 2^b`real`(v) and 2^b`imag`(v) if $r = n+2$. The first n rows of M potentially correspond to relations: whenever the last $r-n$ entries of a column are small. The function `lindep_bit` essentially returns the first column of M truncated to n components.

`GEN lindep_padic(GEN v)` p-adic entries.

`GEN lindep_Xadic(GEN v)` polynomial entries.

`GEN deplin(GEN v)` returns a nonzero kernel vector for a `t_MAT` input.

Deprecated routine:

`GEN lindep2(GEN x, long dig)` analogous to `lindep_bit`, with `dig` counting decimal digits.

13.4.4 Reduction modulo matrices.

`GEN ZC_hnfremdiv(GEN x, GEN y, GEN *Q)` assuming y is an invertible `ZM` in HNF and x is a `ZC`, returns the `ZC` R equal to x mod y (whose i-th entry belongs to $[-y_{i,i}/2, y_{i,i}/2[$). Stack clean *unless* x is already reduced (in which case, returns x itself, not a copy). If Q is not `NULL`, set it to the `ZC` such that $x = yQ + R$.

`GEN ZM_hnfdivrem(GEN x, GEN y, GEN *Q)` reduce each column of the `ZM` x using `ZC_hnfremdiv`. If Q is not `NULL`, set it to the `ZM` such that $x = yQ + R$.

`GEN ZC_hnfrem(GEN x, GEN y)` alias for `ZC_hnfremdiv(x,y,NULL)`.

`GEN ZM_hnfrem(GEN x, GEN y)` alias for `ZM_hnfremdiv(x,y,NULL)`.

`GEN ZC_reducemodmatrix(GEN v, GEN y)` Let y be a ZM, not necessarily square, which is assumed to be LLL-reduced (otherwise, very poor reduction is expected). Size-reduces the ZC v modulo the $\mathbf{Z}$-module Y spanned by y : if the columns of y are denoted by $(y_1, \dots, y_{n-1})$, we return $y_n \equiv v$ modulo Y, such that the Gram-Schmidt coefficients $\mu_{n,j}$ are less than $1/2$ in absolute value for all $j < n$. In short, y_n is almost orthogonal to Y.

`GEN ZM_reducemodmatrix(GEN v, GEN y)` Let y be as in `ZC_reducemodmatrix`, and v be a ZM. This returns a matrix v which is congruent to v modulo the $\mathbf{Z}$-module spanned by y, whose columns are size-reduced. This is faster than repeatedly calling `ZC_reducemodmatrix` on the columns since most of the Gram-Schmidt coefficients can be reused.

`GEN ZC_reducemodlll(GEN v, GEN y)` Let y be an arbitrary ZM, LLL-reduce it then call `ZC_reducemodmatrix`.

`GEN ZM_reducemodlll(GEN v, GEN y)` Let y be an arbitrary ZM, LLL-reduce it then call `ZM_reducemodmatrix`.

Besides the above functions, which were specific to integral input, we also have:

`GEN reducemodinvertible(GEN x, GEN y)` y is an invertible matrix and x a `t_COL` or `t_MAT` of compatible dimension. Returns $x - y\lfloor y^{-1}x\rceil$, which has small entries and differs from x by an integral linear combination of the columns of y. Suitable for `gerepileupto`, but does not collect garbage.

`GEN closemodinvertible(GEN x, GEN y)` returns $x-$`reducemodinvertible`(x, y), i.e. an integral linear combination of the columns of y, which is close to x.

`GEN reducemodlll(GEN x, GEN y)` LLL-reduce the nonsingular `ZM` y and call `reducemodinvertible` to find a small representative of x mod $y\mathbf{Z}^n$. Suitable for `gerepileupto`, but does not collect garbage.

13.5 Finite abelian groups and characters.

13.5.1 Abstract groups.

A finite abelian group G in GP format is given by its Smith Normal Form as a pair $[h, d]$ or triple $[h, d, g]$. Here h is the cardinality of G, (d_i) is the vector of elementary divisors, and (g_i) is a vector of generators. In short, $G = \oplus_{i\le n}(\mathbf{Z}/d_i\mathbf{Z})g_i$, with $d_n \mid \ldots \mid d_2 \mid d_1$ and $\prod d_i = h$.

Let $e(x) := \exp(2i\pi x)$. For ease of exposition, we restrict to complex-valued characters, but everything applies to more general fields K where e denotes a morphism $(\mathbf{Q}, +) \to (K^*, \times)$ such that $e(a/b)$ denotes a b-th root of unity.

A *character* on the abelian group $\oplus(\mathbf{Z}/d_j\mathbf{Z})g_j$ is given by a row vector $\chi = [a_1, \ldots, a_n]$ such that $\chi(\prod g_j^{n_j}) = e(\sum a_j n_j/d_j)$.

`GEN cyc_normalize(GEN d)` shallow function. Given a vector $(d_i)_{i\le n}$ of elementary divisors for a finite group (no d_i vanish), returns the vector $D = [1]$ if $n = 0$ (trivial group) and $[d_1, d_1/d_2, \ldots, d_1/d_n]$ otherwise. This will allow to define characters as $\chi(\prod g_j^{x_j}) = e(\sum_j x_j a_j D_j/D_1)$, see `char_normalize`.

`GEN char_normalize(GEN chi, GEN ncyc)` shallow function. Given a character `chi` $= (a_j)$ and *ncyc* from `cyc_normalize` above, returns the normalized representation $[d, (n_j)]$, such that $\chi(\prod g_j^{x_j}) = \zeta_d^{\sum_j n_j x_j}$, where $\zeta_d = e(1/d)$ and d is *minimal*. In particular, d is the order of `chi`. Shallow function.

`GEN char_simplify(GEN D, GEN N)` given a quasi-normalized character $[D, (N_j)]$ such that $\chi(\prod g_j^{x_j}) = \zeta_D^{\sum_j N_j x_j}$, but where we only assume that D is a multiple of the character order, return a normalized character $[d, (n_j)]$ with d *minimal*. Shallow function.

`GEN char_denormalize(GEN cyc, GEN d, GEN n)` given a normalized representation $[d, n]$ (where d need not be minimal) of a character on the abelian group with abelian divisors `cyc`, return the attached character (where the image of each generator g_i is given in terms of roots of unity of different orders `cyc`$[i]$).

`GEN charconj(GEN cyc, GEN chi)` return the complex conjugate of `chi`.

`GEN charmul(GEN cyc, GEN a, GEN b)` return the product character $a \times b$.

`GEN chardiv(GEN cyc, GEN a, GEN b)` returns the character $a/b = a \times \overline{b}$.

`int char_check(GEN cyc, GEN chi)` return 1 if `chi` is a character compatible with cyclic factors `cyc`, and 0 otherwise.

`GEN cyc2elts(GEN d)` given a `t_VEC` $d = (d_1, \ldots, d_n)$ of nonnegative integers, return the vector of all `t_VECSMALL`s of length n whose i-th entry lies in $[0, d_i[$. Assumes that the product of the d_i fits in a `long`.

`long zv_cyc_minimize(GEN d, GEN c, GEN coprime)` given $d = (d_1, \ldots, d_n)$, $d_n \mid \ldots \mid d_1 \neq 0$ a list of elementary divisors for a finite abelian group as a `t_VECSMALL`, given $c = [g_1, \ldots, g_n]$ representing an element in the group, and given a mask `coprime` (as from `coprimes_zv`(o)) representing a list of forbidden congruence classes modulo o, return an integer k such that `coprime`$[k\%o]$ is nonzero and $k \cdot c$ is lexicographically minimal. For instance, if c is attached to a Dirichlet character χ of order o via the usual identification $\chi(g_i) = \zeta_{g_i}^{c_i}$, then χ^k is a "canonical" representative in the Galois orbit of χ.

`long zv_cyc_minimal(GEN d, GEN c, GEN coprime)` return 1 if `zv_cyc_minimize` would return $k = 1$, i.e. c is already the canonical representative for the attached character orbit.

13.5.2 Dirichlet characters.

The functions in this section are specific to characters on $(\mathbf{Z}/N\mathbf{Z})^*$. The argument G is a special `bid` structure as returned by `znstar0(N, nf_INIT)`. In this case, there are additional ways to input character via Conrey's representation. The character `chi` is either a `t_INT` (Conrey label), a `t_COL` (a Conrey logarithm) or a `t_VEC` (generic character on `bid.gen` as explained in the previous subsection). The following low-level functions are called by GP's generic character functions.

`int zncharcheck(GEN G, GEN chi)` return 1 if `chi` is a valid character and 0 otherwise.

`GEN zncharconj(GEN G, GEN chi)` as `charconj`.

`GEN znchardiv(GEN G, GEN a, GEN b)` as `chardiv`.

`GEN zncharker(GEN G, GEN chi)` as `charker`.

`GEN znchareval(GEN G, GEN chi, GEN n, GEN z)` as `chareval`.

`GEN zncharmul(GEN G, GEN a, GEN b)` as `charmul`.

`GEN zncharpow(GEN G, GEN a, GEN n)` as `charpow`.

`GEN zncharorder(GEN G, GEN chi)` as `charorder`.

The following functions handle characters in Conrey notation (attached to Conrey generators, not `G.gen`):

`int znconrey_check(GEN cyc, GEN chi)` return 1 if `chi` is a valid Conrey logarithm and 0 otherwise.

`GEN znconrey_normalized(GEN G, GEN chi)` return normalized character attached to `chi`, as in `char_normalize` but on Conrey generators.

`GEN znconreyfromchar(GEN G, GEN chi)` return Conrey logarithm attached to the generic (`t_VEC`, on `G.gen`)

`GEN znconreyfromchar_normalized(GEN G, GEN chi)` return normalized Conrey character attached to the generic (`t_VEC`, on `G.gen`) character `chi`.

`GEN znconreylog_normalize(GEN G, GEN m)` given a Conrey logarithm m (`t_COL`), return the attached normalized Conrey character, as in `char_normalize` but on Conrey generators.

GEN znchar_quad(GEN G, GEN D) given a nonzero t_INT D congruent to $0, 1 \bmod 4$, return $(D/.)$ as a character modulo N, given by a Conrey logarithm (t_COL). Assume that $|D|$ divides N.

GEN Zideallog(GEN G, GEN x) return the znconreylog of x expressed on G.gen, i.e. the ordinary discrete logarithm from ideallog.

GEN ncharvecexpo(GEN G, GEN nchi) given nchi $= [d, n]$ a quasi-normalized character (d may be a multiple of the character order), i.e. $\chi(g_i) = e(n[i]/d)$ for all Conrey or SNF generators g_i (as usual, we use SNF generators if n is a t_VEC and the Conrey generators otherwise). Return a t_VECSMALL v such that $v[i] = -1$ if $(i, N) > 1$ else $\chi(i) = e(v[i]/d)$, $1 \le i \le N$.

13.6 Hecke characters.

The functions in this section are specific to Hecke characters. The argument gc is a gchar structure as returned by gcharinit(bnf, mod), and the character chi is a t_COL of components on the SNF generators of gc.

GEN eulerf_gchar(GEN an, GEN p, long prec) an being the first component of a Hecke L-function Ldata (as output by lfungchar) and p a prime number, return the Euler factor at p.

GEN gchari_lfun(GEN gc, GEN chi, GEN w) chi being a t_VEC describing a Hecke character encoded on the internal basis gc[1], return the Ldata structure corresponding to the Hecke L-function associated to chi.

int is_gchar_group(GEN gc) return 1 if gc is a valid gchar structure and 0 otherwise.

GEN lfungchar(GEN gc, GEN chi) return the Ldata structure corresponding to the Hecke L-function associated to chi.

GEN vecan_gchar(GEN an, long n, long prec) an being the first component of a Hecke L-function Ldata (as output by lfungchar), return a t_VEC of length n containing the first n Dirichlet coefficients of this L-function, computed to absolute precision prec.

13.7 Central simple algebras.

13.7.1 Initialization.

Low-level routines underlying alginit; argument rnf (resp. nf) must be true *rnf* (resp. *nf*) structure.

GEN alg_csa_table(GEN nf, GEN mt, long v, long maxord) algebra defined by a multiplication table.

GEN alg_cyclic(GEN rnf, GEN aut, GEN b, long maxord) cyclic algebra $(L/K, \sigma, b)$.

GEN alg_hasse(GEN nf, long d, GEN hi, GEN hf, long v, long maxord) algebra defined by local Hasse invariants.

GEN alg_hilbert(GEN nf, GEN a, GEN b, long v, long maxord) quaternion algebra.

GEN alg_matrix(GEN nf, long n, long v, GEN L, long maxord) matrix algebra.

GEN alg_complete(GEN rnf, GEN aut, GEN hf, GEN hi, long maxord) cyclic algebra $(L/K, \sigma, b)$ with b computed from the Hasse invariants.

GEN alg_changeorder(GEN alg, GEN ord) return the algebra with the integral basis replaced by ord (a t_MAT expressing the basis of the new order in terms of the integral basis of alg). No checks are performed.

13.7.2 Type checks.

`void checkalg(GEN a)` raise an exception if a was not initialized by `alginit`.

`void checklat(GEN al, GEN lat)` raise an exception if `lat` is not a valid full lattice in the algebra `al`.

`void checkhasse(GEN nf, GEN hi, GEN hf, long n)` raise an exception if (`hi`, `hf`) do not describe valid Hasse invariants of a central simple algebra of degree `n` over `nf`.

`long alg_type(GEN al)` internal function called by `algtype`: assume `al` was created by `alginit` (thereby saving a call to `checkalg`). Return values are symbolic rather than numeric:

- `al_NULL`: not a valid algebra.
- `al_TABLE`: table algebra output by `algtableinit`.
- `al_CSA`: central simple algebra output by `alginit` and represented by a multiplication table over its center.
- `al_CYCLIC`: central simple algebra output by `alginit` and represented by a cyclic algebra.

`long alg_model(GEN al, GEN x)` given an element x in algebra al, check for inconsistencies (raise a type error) and return the representation model used for x:

- `al_ALGEBRAIC`: `basistoalg` form, algebraic representation.
- `al_BASIS`: `algtobasis` form, column vector on the integral basis.
- `al_MATRIX`: matrix with coefficients in an algebra.
- `al_TRIVIAL`: trivial algebra of degree 1; can be understood as both basis or algebraic form (since $e_1 = 1$).

13.7.3 Shallow accessors.

All these routines assume their argument was initialized by `alginit` and provide minor speedups compared to the GP equivalent. The routines returning a `GEN` are shallow.

`long alg_get_absdim(GEN al)` low-level version of `algabsdim`.

`long alg_get_dim(GEN al)` low-level version of `algdim`.

`long alg_get_degree(GEN al)` low-level version of `algdegree`.

`GEN alg_get_aut(GEN al)` low-level version of `algaut`.

`GEN alg_get_auts(GEN al)`, given a cyclic algebra $al = (L/K, \sigma, b)$ of degree n, returns the vector of σ^i, $1 \le i < n$.

`GEN alg_get_b(GEN al)` low-level version of `algb`.

`GEN alg_get_basis(GEN al)` low-level version of `algbasis`.

`GEN alg_get_center(GEN al)` low-level version of `algcenter`.

`GEN alg_get_char(GEN al)` low-level version of `algchar`.

`GEN alg_get_hasse_f(GEN al)` low-level version of `alghassef`.

`GEN alg_get_hasse_i(GEN al)` low-level version of `alghassei`.

`GEN alg_get_invbasis(GEN al)` low-level version of `alginvbasis`.

`GEN alg_get_multable(GEN al)` low-level version of `algmultable`.

`GEN alg_get_relmultable(GEN al)` low-level version of `algrelmultable`.

`GEN alg_get_splittingfield(GEN al)` low-level version of `algsplittingfield`.

`GEN alg_get_abssplitting(GEN al)` returns the absolute *nf* structure attached to the *rnf* returned by `algsplittingfield`.

`GEN alg_get_splitpol(GEN al)` returns the relative polynomial defining the *rnf* returned by `algsplittingfield`.

`GEN alg_get_splittingdata(GEN al)` low-level version of `algsplittingdata`.

`GEN alg_get_splittingbasis(GEN al)` the matrix *Lbas* from `algsplittingdata`

`GEN alg_get_splittingbasisinv(GEN al)` the matrix *Lbasinv* from `algsplittingdata`.

`GEN alg_get_tracebasis(GEN al)` returns the traces of the basis elements; used by `algtrace`.

`GEN alglat_get_primbasis(GEN lat)` from the description of `lat` as λL with $L \subset \mathcal{O}_0$ and $\lambda \in \mathbf{Q}$, returns a basis of L.

`GEN alglat_get_scalar(GEN lat)` from the description of `lat` as λL with $L \subset \mathcal{O}_0$ and $\lambda \in \mathbf{Q}$, returns λ.

13.7.4 Other low-level functions.

`GEN conjclasses_algcenter(GEN cc, GEN p)` low-level function underlying `alggroupcenter`, where `cc` is the output of `groupelts_to_conjclasses`, and p is either `NULL` or a prime number. Not stack clean.

`GEN algsimpledec_ss(GEN al, long maps)` assuming that `al` is semisimple, returns the second component of `algsimpledec(al,maps)`.

Chapter 14:
Elliptic curves and arithmetic geometry

This chapter is quite short, but is added as a placeholder, since we expect the library to expand in that direction.

14.1 Elliptic curves.

Elliptic curves are represented in the Weierstrass model

$$(E) : y^2z + a_1xyz + a_3yz = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,$$

by the 5-tuple $[a_1, a_2, a_3, a_4, a_6]$. Points in the projective plane are represented as follows: the point at infinity $(0 : 1 : 0)$ is coded as `[0]`, a finite point $(x : y : 1)$ outside the projective line at infinity $z = 0$ is coded as $[x, y]$. Note that other points at infinity than $(0 : 1 : 0)$ cannot be represented; this is harmless, since they do not belong to any of the elliptic curves E above.

Points on the curve are just projective points as described above, they are not tied to a curve in any way: the same point may be used in conjunction with different curves, provided it satisfies their equations (if it does not, the result is usually undefined). In particular, the point at infinity belongs to all elliptic curves.

As with `factor` for polynomial factorization, the 5-tuple $[a_1, a_2, a_3, a_4, a_6]$ implicitly defines a base ring over which the curve is defined. Point coordinates must be operation-compatible with this base ring (`gadd`, `gmul`, `gdiv` involving them should not give errors).

14.1.1 Types of elliptic curves.

We call a 5-tuble as above an `ell5`; most functions require an `ell` structure, as returned by `ellinit`, which contains additional data (usually dynamically computed as needed), depending on the base field.

`GEN ellinit(GEN E, GEN D, long prec)`, returns an `ell` structure, attached to the elliptic curve E : either an `ell5`, a pair $[a_4, a_6]$ or a `t_STR` in Cremona's notation, e.g. `"11a1"`. The optional D (`NULL` to omit) describes the domain over which the curve is defined.

14.1.2 Type checking.

`void checkell(GEN e)` raise an error unless e is an *ell*.

`int checkell_i(GEN e)` return 1 if e is an *ell* and 0 otherwise.

`void checkell5(GEN e)` raise an error unless e is an *ell* or an *ell5*.

`void checkellpt(GEN z)` raise an error unless z is a point (either finite or at infinity).

`long ell_get_type(GEN e)` returns the domain type over which the curve is defined, one of

`t_ELL_Q` the field of rational numbers;

`t_ELL_NF` a number field;

`t_ELL_Qp` the field of p-adic numbers, for some prime p;

`t_ELL_Fp` a prime finite field, base field elements are represented as `Fp`, i.e. a `t_INT` reduced modulo p;

`t_ELL_Fq` a nonprime finite field (a prime finite field can also be represented by this subtype, but this is inefficient), base field elements are represented as `t_FFELT`;

`t_ELL_Rg` none of the above.

`void checkell_Fq(GEN e)` checks whether e is an `ell`, defined over a finite field (either prime or nonprime). Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Q(GEN e)` checks whether e is an `ell`, defined over $\mathbf{Q}$. Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Qp(GEN e)` checks whether e is an `ell`, defined over some $\mathbf{Q}_p$. Otherwise the function raises a `pari_err_TYPE` exception.

`void checkellisog(GEN v)` raise an error unless v is an isogeny, from `ellisogeny`.

14.1.3 Extracting info from an `ell` structure.

These functions expect an `ell` argument. If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

14.1.3.1 All domains.

`GEN ell_get_a1(GEN e)`

`GEN ell_get_a2(GEN e)`

`GEN ell_get_a3(GEN e)`

`GEN ell_get_a4(GEN e)`

`GEN ell_get_a6(GEN e)`

`GEN ell_get_b2(GEN e)`

`GEN ell_get_b4(GEN e)`

`GEN ell_get_b6(GEN e)`

`GEN ell_get_b8(GEN e)`

`GEN ell_get_c4(GEN e)`

`GEN ell_get_c6(GEN e)`

`GEN ell_get_disc(GEN e)`

`GEN ell_get_j(GEN e)`

14.1.3.2 Curves over $\mathbf{Q}$.

`GEN ellQ_get_N(GEN e)` returns the curve conductor

`void ellQ_get_Nfa(GEN e, GEN *N, GEN *faN)` sets N to the conductor and `faN` to its factorization

`int ell_is_integral(GEN e)` return 1 if e is given by an integral model, and 0 otherwise.

`long ellQ_get_CM(GEN e)` if e has CM by a principal imaginary quadratic order, return its discriminant. Else return 0.

`long ellap_CM_fast(GEN e, ulong p, long CM)` assuming that p does not divide the discriminant of E (in particular, E has good reduction at p), and that `CM` is as given by `ellQ_get_CM`, return the trace of Frobenius for $E/\mathbf{F}_p$. This is meant to quickly compute lots of a_p, esp. when e has CM by a principal quadratic order.

`long ellrootno_global(GEN e)` returns the global root number $c \in \{-1, 1\}$.

`GEN ellheightoo(GEN E, GEN P, long prec)` given $P = [x, y]$ an affine point on E, return

$$\lambda_\infty(P) + \frac{1}{12}\log|\mathrm{disc}E| = \frac{1}{2}\mathrm{real}(z\eta(z)) - \log|\sigma(E, z)| \in \mathbf{R},$$

where $\lambda_\infty(P)$ is the canonical local height at infinity and z is `ellpointtoz`(E, P). This is computed using Mestre's (quadratically convergent) AGM algorithm.

`long ellorder_Q(GEN E, GEN P)` return the order of $P \in E(\mathbf{Q})$, using the impossible value 0 for a point of infinite order. Ultimately called by the generic `ellorder` function.

`GEN point_to_a4a6(GEN E, GEN P, GEN p, GEN *a4)` given $E/\mathbf{Q}$, $p \neq 2, 3$ not dividing the discriminant of E and $P \in E(\mathbf{Q})$ outside the kernel of reduction, return the image of P on the short Weierstrass model $y^2 = x^3 + a_4x + a_6$ isomorphic to the reduction E_p of E at p. Also set `a4` to the a_4 coefficient in the above model. This function allows quick computations modulo varying primes p, avoiding the overhead of `ellinit`(E, p), followed by a change of coordinates. It produces data suitable for `FpE` routines.

`GEN point_to_a4a6_Fl(GEN E, GEN P, ulong p, ulong *pa4)` as `point_to_a4a6`, returning a `Fle`.

`GEN elldatagenerators(GEN E)` returns generators for $E(\mathbf{Q})$ extracted from Cremona's table.

`GEN ellanal_globalred(GEN e, GEN *v)` takes an *ell* over $\mathbf{Q}$ and returns a global minimal model E (in `ellinit` form, over $\mathbf{Q}$) for e suitable for analytic computations related to the curve L series: it contains `ellglobalred` data, as well as global and local root numbers. If `v` is not `NULL`, set `*v` to the needed change of variable: `NULL` if e was already the standard minimal model, such that $E =$ `ellchangecurve(e, v)` otherwise. Compared to the direct use of `ellchangecurve` followed by `ellrootno`, this function avoids converting unneeded dynamic data and avoids potential memory leaks (the changed curve would have had to be deleted using `obj_free`). The original curve e is updated as well with the same information.

`GEN ellanal_globalred_all(GEN e, GEN *v, GEN *N, GEN *tam)` as `ellanal_globalred`; further set `*N` to the curve conductor and `*tam` to the product of the local Tamagawa numbers, including the factor at infinity (multiply by the number of connected components of $e(\mathbf{R})$).

`GEN ellintegralmodel(GEN e, GEN *pv)` return an integral model for e (in `ellinit` form, over $\mathbf{Q}$). Set $v =$ `NULL` (already integral, we returned e itself), else to the variable change $[u, 0, 0, 0]$ making e integral. We have $u = 1/t$, $t > 1$.

GEN ellintegralmodel_i(GEN e, GEN *pv) shallow version of ellintegralmodel.

GEN ellQtwist_bsdperiod(GEN E, long s) let E be a rational elliptic curve given by a minimal model, Λ_E its period lattice, and $s \in \{-1,1\}$. Let $\Omega_E^{\pm}$ be the canonical periods in $\sqrt{\pm 1}\mathbf{R}^+$ generating $\Lambda_E \cap \sqrt{\pm 1}\mathbf{R}$. Return Ω_E^+ if $s = 1$ and Ω_E^- if $s = -1$.

GEN elltors_psylow(GEN e, ulong p) as elltors, but return the p-Sylow subgroup of the torsion group.

GEN elleulerf(GEN E, GEN p) returns the Euler factor at p of the L-function associated to the curve E defined over a number field.

Deprecated routines.

GEN elltors0(GEN e, long flag) this function is deprecated; use elltors

14.1.3.3 Curves over a number field *nf*.

Let K be the number field over which E is defined, given by a *nf* or *bnf* structure.

GEN ellnf_get_nf(GEN E) returns the underlying nf.

GEN ellnf_get_bnf(GEN x) returns NULL if K does not contain a *bnf* structure, else return the *bnf*.

GEN ellnf_vecarea(GEN E) returns the vector of the period lattices areas of all the complex embeddings of E in the same order as E.nf.roots.

GEN ellnf_veceta(GEN E) returns the vector of the quasi-periods of all the complex embeddings of E in the same order as E.nf.roots.

GEN ellnf_vecomega(GEN E) returns the vector of the periods of all the complex embeddings of E in the same order as E.nf.roots.

14.1.3.4 Curves over $\mathbf{Q}_p$.

GEN ellQp_get_p(GEN E) returns p

long ellQp_get_prec(GEN E) returns the default p-adic accuracy to which we must compute approximate results attached to E.

GEN ellQp_get_zero(GEN x) returns $O(p^n)$, where n is the default p-adic accuracy as above.

The following functions are only defined when E has multiplicative reduction (Tate curves):

GEN ellQp_Tate_uniformization(GEN E, long prec) returns a t_VEC containing $u^2, u, q, [a, b]$, at p-adic precision prec.

GEN ellQp_u(GEN E, long prec) returns u.

GEN ellQp_u2(GEN E, long prec) returns u^2.

GEN ellQp_q(GEN E, long prec) returns the Tate period q.

GEN ellQp_ab(GEN E, long prec) returns $[a, b]$.

GEN ellQp_AGM(GEN E, long prec) returns $[a, b, R, v]$, where v is an integer, a, b, R are vectors describing the sequence of 2-isogenous curves $E_i : y^2 = x(x + A_i)(x + A_i - B_i)$, $i \geq 1$ converging to the singular curve $E_\infty : y^2 = x^2(x + M)$. We have $a[i] = A[i]p^v$, $b[i] = B[i]p^v$, $R[i] = A_i - B_i$. These are used in ellpointtoz and ellztopoint.

GEN ellQp_L(GEN E, long prec) returns the $\mathcal{L}$-invariant L.

GEN ellQp_root(GEN E, long prec) returns e_1.

14.1.3.5 Curves over a finite field $\mathbf{F}_q$.

`GEN ellff_get_p(GEN E)` returns the characteristic

`GEN ellff_get_field(GEN E)` returns p if $\mathbf{F}_q$ is a prime field, and a `t_FFELT` belonging to $\mathbf{F}_q$ otherwise.

`GEN ellff_get_card(GEN E)` returns $\#E(\mathbf{F}_q)$

`GEN ellff_get_gens(GEN E)` returns a minimal set of generators for $E(\mathbf{F}_q)$.

`GEN ellff_get_group(GEN E)` returns `ellgroup`(E).

`GEN ellff_get_m(GEN E)` returns the `t_INT` m as needed by the `gen_ellgroup` function (the order of the pairing required to verify a generating set).

`GEN ellff_get_o(GEN E)` returns $[d, \texttt{factord}]$, where d is the exponent of $E(\mathbf{F}_q)$.

`GEN ellff_get_D(GEN E)` returns the elementary divisors for $E(\mathbf{F}_q)$ in a form suitable for `gen_ellgens`: either $[d_1]$ or $[d_1, d_2]$, where d_1 is in `elff_get_o` format.

$[d, \texttt{factord}]$, where d is the exponent of $E(\mathbf{F}_q)$.

`GEN ellff_get_a4a6(GEN E)` returns a canonical "short model" for E, and the corresponding change of variable $[u, r, s, t]$. For $p \neq 2, 3$, this is $[A_4, A_6, [u, r, s, t]]$, corresponding to $y^2 = x^3 + A_4x + A_6$, where $A_4 = -27c_4$, $A_6 = -54c_6$, $[u, r, s, t] = [6, 3b_2, 3a_1, 108a_3]$.

- If $p = 3$ and the curve is ordinary ($b_2 \neq 0$), this is $[[b_2], A_6, [1, v, -a_1, -a_3]]$, corresponding to

$$y^2 = x^3 + b_2x^2 + A_6,$$

where $v = b_4/b_2$, $A_6 = b_6 - v(b_4 + v^2)$.

- If $p = 3$ and the curve is supersingular ($b_2 = 0$), this is $[-b_4, b_6, [1, 0, -a_1, -a_3]]$, corresponding to

$$y^2 = x^3 + 2b_4x + b_6.$$

- If $p = 2$ and the curve is ordinary ($a_1 \neq 0$), return $[A_2, A_6, [a_1^{-1}, da_1^{-2}, 0, (a_4 + d^2)a_1^{-1}]]$, corresponding to

$$y^2 + xy = x^3 + A_2x^2 + A_6,$$

where $d = a_3/a_1$, $a_1^2A_2 = (a_2 + d)$ and

$$a_1^6A_6 = d^3 + a_2d^2 + a_4d + a_6 + (a_4^2 + d^4)a_1^{-2}.$$

- If $p = 2$ and the curve is supersingular ($a_1 = 0$, $a_3 \neq 0$), return $[[a_3, A_4, 1/a_3], A_6, [1, a_2, 0, 0]]$, corresponding to

$$y^2 + a_3y = x^3 + A_4x + A_6,$$

where $A_4 = a_2^2 + a_4$, $A_6 = a_2a_4 + a_6$. The value $1/a_3$ is included in the vector since it is frequently needed in computations.

14.1.3.6 Curves over C. (This includes curves over **Q**!)

`long ellR_get_prec(GEN E)` return the default accuracy to which we must compute approximate results attached to E.

`GEN ellR_ab(GEN E, long prec)` return $[a, b]$

`GEN ellR_omega(GEN x, long prec)` return periods $[\omega_1, \omega_2]$.

`GEN ellR_eta(GEN E, long prec)` return quasi-periods $[\eta_1, \eta_2]$.

`GEN ellR_area(GEN x, long prec)` return the area $(\Im(\omega_1\overline{\omega_2}))$.

`GEN ellR_roots(GEN E, long prec)` return $[e_1, e_2, e_3]$. If E is defined over **R**, then e_1 is real. If furthermore disc$E > 0$, then $e_1 > e_2 > e_3$.

`long ellR_get_sign(GEN E)` if E is defined over **R** returns the signe of its discriminant, otherwise return 0.

14.1.4 Points.

`int ell_is_inf(GEN z)` tests whether the point z is the point at infinity.

`GEN ellinf()` returns the point at infinity `[0]`.

14.1.5 Change of variables.

`GEN ellchangeinvert(GEN w)` given a change of variables $w = [u, r, s, t]$, returns the inverse change of variables w', such that if $E' =$ `ellchangecurve(E, w)`, then $E =$ `ellchangecurve`(E, w').

14.1.6 Generic helper functions.

The naming scheme assumes an affine equation $F(x, y) = f(x) - (y^2 + h(x)y) = 0$ in standard Weierstrass form: $f = x^3 + a_2x^2 + a_4x + a_6$, $h = a_1x + a_3$. Unless mentionned otherwise, these routine assume that all arguments are compatible with generic functions of `gadd` or `gmul` type. In particular they do not handle elements in number field in `nfalgtobasis` format.

`GEN ellbasechar(GEN E)` returns the characteristic of the base ring over which E is defined.

`GEN ec_bmodel(GEN E)` returns the polynomial $4x^3 + b_2x^2 + 2b_4x + b_6$.

`GEN ec_phi2(GEN E)` returns the polynomial $x^4 - b_4x^2 - 2b_6 * X - b_8$.

`GEN ec_f_evalx(GEN E, GEN x)` returns $f(x)$.

`GEN ec_h_evalx(GEN E, GEN x)` returns $h(x)$.

`GEN ec_dFdx_evalQ(GEN E, GEN Q)` returns $3x^2 + 2a_2x + a_4 - a_1y$, where $Q = [x, y]$.

`GEN ec_dFdy_evalQ(GEN E, GEN Q)` returns $-(2y + a_1x + a_3)$, where $Q = [x, y]$.

`GEN ec_dmFdy_evalQ(GEN e, GEN Q)` returns $2y + a_1x + a_3$, where $Q = [x, y]$.

`GEN ec_2divpol_evalx(GEN E, GEN x)` returns $4x^3 + b_2x^2 + 2b_4x + b_6$. This function supports inputs in `nfalgtobasis` format.

`GEN ec_half_deriv_2divpol_evalx(GEN E, GEN x)` returns $6x^2 + b_2x + b_4$.

`GEN ec_3divpol_evalx(GEN E, GEN x)` returns $3x^4 + b_2x^2 + 3b_4x^2 + 3b_6x + b_8$.

14.1.7 Functions to handle elliptic curves over finite fields.

14.1.7.1 Tolerant routines.

`GEN ellap(GEN E, GEN p)` given a prime number p and an elliptic curve defined over $\mathbf{Q}$ or $\mathbf{Q}_p$ (assumed integral and minimal at p), computes the trace of Frobenius $a_p = p + 1 - \#E(\mathbf{F}_p)$. If E is defined over a nonprime finite field $\mathbf{F}_q$, ignore p and return $q + 1 - \#E(\mathbf{F}_q)$. When p is implied (E defined over $\mathbf{Q}_p$ or a finite field), p can be omitted (set to `NULL`).

14.1.7.2 Curves defined a nonprime finite field. In this subsection, we assume that `ell_get_type`(E) is `t_ELL_Fq`. (As noted above, a curve defined over $\mathbf{Z}/p\mathbf{Z}$ can be represented as a `t_ELL_Fq`.)

`GEN FF_elltwist(GEN E)` returns the coefficients $[a_1, a_2, a_3, a_4, a_6]$ of the quadratic twist of E.

`GEN FF_ellmul(GEN E, GEN P, GEN n)` returns $[n]P$ where n is an integer and P is a point on the curve E.

`GEN FF_ellrandom(GEN E)` returns a random point in $E(\mathbf{F}_q)$. This function never returns the point at infinity, unless this is the only point on the curve.

`GEN FF_ellorder(GEN E, GEN P, GEN o)` returns the order of the point P, where o is a multiple of the order of P, or its factorization.

`GEN FF_ellcard(GEN E)` returns $\#E(\mathbf{F}_q)$.

`GEN FF_ellcard_SEA(GEN E, long s)` This function returns $\#E(\mathbf{F}_q)$, using the Schoof-Elkies-Atkin algorithm. Assume $p \neq 2, 3$. The parameter s has the same meaning as in `Fp_ellcard_SEA`.

`GEN FF_ellgens(GEN E)` returns the generators of the group $E(\mathbf{F}_q)$.

`GEN FF_elllog(GEN E, GEN P, GEN G, GEN o)` Let `G` be a point of order `o`, return e such that $[e]P = G$. If e does not exists, the result is undefined.

`GEN FF_ellgroup(GEN E, GEN *pm)` returns the structure of the Abelian group $E(\mathbf{F}_q)$ and set `*pm` to m (see `gen_ellgens`).

`GEN FF_ellweilpairing(GEN E, GEN P, GEN Q, GEN m)` returns the Weil pairing of the points of m-torsion P and Q.

`GEN FF_elltatepairing(GEN E, GEN P, GEN Q, GEN m)` returns the Tate pairing of P and Q, where $[m]P = 0$.

14.2 Arithmetic on elliptic curve over a finite field in simple form.

The functions in this section no longer operate on elliptic curve structures, as seen up to now. They are used to implement those higher-level functions without using cached information and thus require suitable explicitly enumerated data.

14.2.1 Helper functions.

`GEN elltrace_extension(GEN t, long n, GEN q)` Let E some elliptic curve over $\mathbf{F}_q$ such that the trace of the Frobenius is t, returns the trace of the Frobenius over $\mathbf{F}_q^n$.

14.2.2 Elliptic curves over $\mathbf{F}_p$, $p > 3$.

Let p a prime number and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in $\mathbf{F}_p$. A `FpE` is a point of $E(\mathbf{F}_p)$. Since an affine point and a_4 determine an unique $a6$, most functions do not take a_6 as an argument. A `FpE` is either the point at infinity (`ellinf()`) or a FpV whith two components. The parameters a_4 and a_6 are given as `t_INT`s when required.

`GEN Fp_ellj(GEN a4, GEN a6, GEN p)` returns the j-invariant of the curve E.

`int Fp_elljissupersingular(GEN j, GEN p)` returns 1 if j is the j-invariant of a supersingular curve over $\mathbf{F}_p$, 0 otherwise.

`GEN Fp_ellcard(GEN a4, GEN a6, GEN p)` returns the cardinality of the group $E(\mathbf{F}_p)$.

`GEN Fp_ellcard_SEA(GEN a4, GEN a6, GEN p, long s)` This function returns $\#E(\mathbf{F}_p)$, using the Schoof-Elkies-Atkin algorithm. If the `seadata` package is installed, the function will be faster.

The extra flag `s`, if set to a nonzero value, causes the computation to return `gen_0` (an impossible cardinality) if one of the small primes ℓ divides the curve order but does not divide s. For cryptographic applications, where one is usually interested in curves of prime order, setting $s = 1$ efficiently weeds out most uninteresting curves; if curves of order a power of 2 times a prime are acceptable, set $s = 2$. If moreover `s` is negative, similar checks are performed for the twist of the curve.

`GEN Fp_ffellcard(GEN a4, GEN a6, GEN q, long n, GEN p)` returns the cardinality of the group $E(\mathbf{F}_q)$ where $q = p^n$.

`GEN Fp_ellgroup(GEN a4, GEN a6, GEN N, GEN p, GEN *pm)` returns the group structure D of the group $E(\mathbf{F}_p)$, which is assumed to be of order N and set `*pm` to m.

`GEN Fp_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN p)` returns generators of the group $E(\mathbf{F}_p)$ with the base change `ch` (see `FpE_changepoint`), where D and m are as returned by `Fp_ellgroup`.

`GEN Fp_elldivpol(GEN a4, GEN a6, long n, GEN p)` returns the n-division polynomial of the elliptic curve E.

`void Fp_elltwist(GEN a4, GEN a6, GEN p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of E.

14.2.3 `FpE`.

`GEN FpE_add(GEN P, GEN Q, GEN a4, GEN p)` returns the sum $P + Q$ in the group $E(\mathbf{F}_p)$, where E is defined by $E : y^2 = x^3 + a_4x + a_6$, for any value of a_6 compatible with the points given.

`GEN FpE_sub(GEN P, GEN Q, GEN a4, GEN p)` returns $P - Q$.

`GEN FpE_dbl(GEN P, GEN a4, GEN p)` returns $2.P$.

`GEN FpE_neg(GEN P, GEN p)` returns $-P$.

`GEN FpE_mul(GEN P, GEN n, GEN a4, GEN p)` return $n.P$.

`GEN FpE_changepoint(GEN P, GEN m, GEN a4, GEN p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FpV`).

`GEN FpE_changepointinv(GEN P, GEN m, GEN a4, GEN p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FpV`).

GEN random_FpE(GEN a4, GEN a6, GEN p) returns a random point on $E(\mathbf{F}_p)$, where E is defined by $E : y^2 = x^3 + a_4x + a_6$.

GEN FpE_order(GEN P, GEN o, GEN a4, GEN p) returns the order of P in the group $E(\mathbf{F}_p)$, where o is a multiple of the order of P, or its factorization.

GEN FpE_log(GEN P, GEN G, GEN o, GEN a4, GEN p) Let G be a point of order o, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

GEN FpE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN p) returns the Tate pairing of the point of m-torsion P and the point Q.

GEN FpE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN p) returns the Weil pairing of the points of m-torsion P and Q.

GEN FpE_to_mod(GEN P, GEN p) returns P as a vector of t_INTMODs.

GEN RgE_to_FpE(GEN P, GEN p) returns the FpE obtained by applying Rg_to_Fp coefficientwise.

14.2.4 Fle. Let p be a prime ulong, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are ulong. A Fle is either the point at infinity (ellinf()), or a Flv with two components $[x, y]$.

long Fl_elltrace(ulong a4, ulong a6, ulong p) returns the trace t of the Frobenius of $E(\mathbf{F}_p)$. The cardinality of $E(\mathbf{F}_p)$ is thus $p + 1 - t$, which might not fit in an ulong.

long Fl_elltrace_CM(long CM, ulong a4, ulong a6, ulong p) as Fl_elltrace. If CM is 0, use the standard algorithm; otherwise assume the curve has CM by a principal imaginary quadratic order of discriminant CM and use a faster algorithm. Useful when the curve is the reduction of $E/\mathbf{Q}$, which has CM by a principal order, and we need the trace of Frobenius for many distinct p, see ellQ_get_CM.

ulong Fl_elldisc(ulong a4, ulong a6, ulong p) returns the discriminant of the curve E.

ulong Fl_elldisc_pre(ulong a4, ulong a6, ulong p, ulong pi) returns the discriminant of the curve E, assuming pi is the pseudoinverse of p.

ulong Fl_ellj(ulong a4, ulong a6, ulong p) returns the j-invariant of the curve E.

ulong Fl_ellj_pre(ulong a4, ulong a6, ulong p, ulong pi) returns the j-invariant of the curve E, assuming pi is the pseudoinverse of p.

void Fl_ellj_to_a4a6(ulong j, ulong p, ulong *pa4, ulong *pa6) sets *pa4 to a_4 and *pa6 to a_6 where a_4 and a_6 define a fixed elliptic curve with j-invariant j.

void Fl_elltwist(ulong a4, ulong a6, ulong p, ulong *pA4, ulong *pA6) set *pA4 to A_4 and *pA6 to A_6 where A_4 and A_6 define the twist of E.

void Fl_elltwist_disc(ulong a4, ulong a6, ulong D, ulong p, ulong *pA4, ulong *pA6) sets *pA4 to A_4 and *pA6 to A_6 where A_4 and A_6 define the twist of E by the discriminant D.

GEN Fl_ellptors(ulong l, ulong N, ulong a4, ulong a6, ulong p) return a basis of the l-torsion subgroup of E.

GEN Fle_add(GEN P, GEN Q, ulong a4, ulong p)

GEN Fle_dbl(GEN P, ulong a4, ulong p)

GEN Fle_sub(GEN P, GEN Q, ulong a4, ulong p)

GEN Fle_mul(GEN P, GEN n, ulong a4, ulong p)

GEN Fle_mulu(GEN P, ulong n, ulong a4, ulong p)

GEN Fle_order(GEN P, GEN o, ulong a4, ulong p)

GEN Fle_log(GEN P, GEN G, GEN o, ulong a4, ulong p)

GEN Fle_tatepairing(GEN P, GEN Q, ulong m, ulong a4, ulong p)

GEN Fle_weilpairing(GEN P, GEN Q, ulong m, ulong a4, ulong p)

GEN random_Fle(ulong a4, ulong a6, ulong p)

GEN random_Fle_pre(ulong a4, ulong a6, ulong p, ulong pi)

GEN Fle_changepoint(GEN x, GEN ch, ulong p), ch is assumed to give the change of coordinates $[u, r, s, t]$ as a t_VECSMALL.

GEN Fle_changepointinv(GEN x, GEN ch, ulong p), as Fle_changepoint

14.2.5 FpJ.

Let $p > 3$ be a prime t_INT, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4 \times x + a_6$, where a_4 and a_6 are t_INT. A FpJ is a FpV with three components $[x, y, z]$, representing the affine point $[x/z^2, y/z^3]$ in Jacobian coordinates, the point at infinity being represented by $[1, 1, 0]$. The following must holds: $y^2 = x^3 + a_4xz^4 + a_6z^6$. For all nonzero u, the points $[u^2x, u^3y, uz]$ and $[x, y, z]$ are representing the same affine point.

GEN FpJ_add(GEN P, GEN Q, GEN a4, GEN p)

GEN FpJ_dbl(GEN P, GEN a4, GEN p)

GEN FpJ_mul(GEN P, GEN n, GEN a4, GEN p);

GEN FpJ_neg(GEN P, GEN p) return $-P$.

GEN FpJ_to_FpE(GEN P, GEN p) return the corresponding FpE.

GEN FpE_to_FpJ(GEN P) return the corresponding FpJ.

14.2.6 Flj.

Let $p > 3$ be a prime. Below, pi is assumed to be the pseudoinverse of p (see get_Fl_red).

GEN Fle_to_Flj(GEN P) convert a Fle to an equivalent Flj.

GEN Flj_to_Fle(GEN P, ulong p) convert a Flj to the equivalent Fle.

GEN Flj_to_Fle_pre(GEN P, ulong p, ulong pi) convert a Flj to the equivalent Fle.

GEN Flj_add_pre(GEN P, GEN Q, ulong a4, ulong p, ulong pi)

GEN Flj_dbl_pre(GEN P, ulong a4, ulong p, ulong pi)

GEN Flj_neg(GEN P, ulong p) return $-P$.

GEN Flj_mulu_pre(GEN P, ulong n, ulong a4, ulong p, ulong pi)

GEN random_Flj_pre(ulong a4, ulong a6, ulong p, ulong pi)

GEN Flj_changepointinv_pre(GEN P, GEN ch, ulong p, ulong pi) where ch is the Flv $[u, r, s, t]$.

GEN FljV_factorback_pre(GEN P, GEN L, ulong p, ulong pi)

14.2.7 Elliptic curves over $\mathbf{F}_{2^n}$. Let T be an irreducible F2x and E the elliptic curve given by either the equation $E : y^2 + x * y = x^3 + a_2x^2 + a_6$, where a_2, a_6 are F2x in $\mathbf{F}_2[X]/(T)$ (ordinary case) or $E : y^2 + a_3 * y = x^3 + a_4x + a_6$, where a_3, a_4, a_6 are F2x in $\mathbf{F}_2[X]/(T)$ (supersingular case).

A F2xqE is a point of $E(\mathbf{F}_2[X]/(T))$. In the supersingular case, the parameter a2 is actually the t_VEC $[a_3, a_4, a_3^{-1}]$.

GEN F2xq_ellcard(GEN a2, GEN a6, GEN T) Return the order of the group $E(\mathbf{F}_2[X]/(T))$.

GEN F2xq_ellgroup(GEN a2, GEN a6, GEN N, GEN T, GEN *pm) Return the group structure D of the group $E(\mathbf{F}_2[X]/(T))$, which is assumed to be of order N and set *pm to m.

GEN F2xq_ellgens(GEN a2, GEN a6, GEN ch, GEN D, GEN m, GEN T) Returns generators of the group $E(\mathbf{F}_2[X]/(T))$ with the base change ch (see F2xqE_changepoint), where D and m are as returned by F2xq_ellgroup.

void F2xq_elltwist(GEN a4, GEN a6, GEN T, GEN *a4t, GEN *a6t) sets *a4t and *a6t to the parameters of the quadratic twist of E.

14.2.8 F2xqE.

GEN F2xqE_changepoint(GEN P, GEN m, GEN a2, GEN T) returns the image Q of the point P on the curve $E : y^2 + x * y = x^3 + a_2x^2 + a_6$ by the coordinate change m (which is a F2xqV).

GEN F2xqE_changepointinv(GEN P, GEN m, GEN a2, GEN T) returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a F2xqV).

GEN F2xqE_add(GEN P, GEN Q, GEN a2, GEN T)

GEN F2xqE_sub(GEN P, GEN Q, GEN a2, GEN T)

GEN F2xqE_dbl(GEN P, GEN a2, GEN T)

GEN F2xqE_neg(GEN P, GEN a2, GEN T)

GEN F2xqE_mul(GEN P, GEN n, GEN a2, GEN T)

GEN random_F2xqE(GEN a2, GEN a6, GEN T)

GEN F2xqE_order(GEN P, GEN o, GEN a2, GEN T) returns the order of P in the group $E(\mathbf{F}_2[X]/(T))$, where o is a multiple of the order of P, or its factorization.

GEN F2xqE_log(GEN P, GEN G, GEN o, GEN a2, GEN T) Let G be a point of order o, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

GEN F2xqE_tatepairing(GEN P, GEN Q, GEN m, GEN a2, GEN T) returns the Tate pairing of the point of m-torsion P and the point Q.

GEN F2xqE_weilpairing(GEN Q, GEN Q, GEN m, GEN a2, GEN T) returns the Weil pairing of the points of m-torsion P and Q.

GEN RgE_to_F2xqE(GEN P, GEN T) returns the F2xqE obtained by applying Rg_to_F2xq coefficien-twise.

14.2.9 Elliptic curves over $\mathbf{F}_q$, small characteristic $p > 2$. Let $p > 2$ be a prime `ulong`, T an irreducible `Flx` mod p, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are `Flx` in $\mathbf{F}_p[X]/(T)$. A `FlxqE` is a point of $E(\mathbf{F}_p[X]/(T))$.

In the special case $p = 3$, ordinary elliptic curves ($j(E) \neq 0$) cannot be represented as above, but admit a model $E : y^2 = x^3 + a_2x^2 + a_6$ with a_2 and a_6 being `Flx` in $\mathbf{F}_3[X]/(T)$. In that case, the parameter `a2` is actually stored as a `t_VEC`, $[a_2]$, to avoid ambiguities.

`GEN Flxq_ellj(GEN a4, GEN a6, GEN T, ulong p)` returns the j-invariant of the curve E.

`void Flxq_ellj_to_a4a6(GEN j, GEN T, ulong p, GEN *pa4, GEN *pa6)` sets `*pa4` to a_4 and `*pa6` to a_6 where a_4 and a_6 define a fixed elliptic curve with j-invariant j.

`GEN Flxq_ellcard(GEN a4, GEN a6, GEN T, ulong p)` returns the order of $E(\mathbf{F}_p[X]/(T))$.

`GEN Flxq_ellgroup(GEN a4, GEN a6, GEN N, GEN T, ulong p, GEN *pm)` returns the group structure D of the group $E(\mathbf{F}_p[X]/(T))$, which is assumed to be of order N and sets `*pm` to m.

`GEN Flxq_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, ulong p)` returns generators of the group $E(\mathbf{F}_p[X]/(T))$ with the base change `ch` (see `FlxqE_changepoint`), where D and m are as returned by `Flxq_ellgroup`.

`void Flxq_elltwist(GEN a4, GEN a6, GEN T, ulong p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of E.

14.2.10 `FlxqE`.

Let $p > 2$ be a prime number.

`GEN FlxqE_changepoint(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FlxqV`).

`GEN FlxqE_changepointinv(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FlxqV`).

`GEN FlxqE_add(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_sub(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_dbl(GEN P, GEN a4, GEN T, ulong p)`

`GEN FlxqE_neg(GEN P, GEN T, ulong p)`

`GEN FlxqE_mul(GEN P, GEN n, GEN a4, GEN T, ulong p)`

`GEN random_FlxqE(GEN a4, GEN a6, GEN T, ulong p)`

`GEN FlxqE_order(GEN P, GEN o, GEN a4, GEN T, ulong p)` returns the order of P in the group $E(\mathbf{F}_p[X]/(T))$, where o is a multiple of the order of P, or its factorization.

`GEN FlxqE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, ulong p)` Let `G` be a point of order `o`, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

`GEN FlxqE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Tate pairing of the point of m-torsion P and the point Q.

`GEN FlxqE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Weil pairing of the points of m-torsion P and Q.

GEN FlxqE_weilpairing_pre(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p, ulong pi)
, where pi is a pseudoinverse of p, or 0 in which case we assume SMALL_ULONG(p).

GEN RgE_to_FlxqE(GEN P, GEN T, ulong p) returns the FlxqE obtained by applying Rg_to_Flxq coefficientwise.

14.2.11 Elliptic curves over $\mathbf{F}_q$, large characteristic .

Let $p > 3$ be a prime number, T an irreducible polynomial mod p, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$ with a_4 and a_6 in $\mathbf{F}_p[X]/(T)$. A FpXQE is a point of $E(\mathbf{F}_p[X]/(T))$.

GEN FpXQ_ellj(GEN a4, GEN a6, GEN T, GEN p) returns the j-invariant of the curve E.

int FpXQ_elljissupersingular(GEN j, GEN T, GEN p) returns 1 if j is the j-invariant of a supersingular curve over $\mathbf{F}_p[X]/(T)$, 0 otherwise.

GEN FpXQ_ellcard(GEN a4, GEN a6, GEN T, GEN p) returns the order of $E(\mathbf{F}_p[X]/(T))$.

GEN Fq_ellcard_SEA(GEN a4, GEN a6, GEN q, GEN T, GEN p, long s) This function returns $\#E(\mathbf{F}_p[X]/(T))$, using the Schoof-Elkies-Atkin algorithm. Assume $p \neq 2, 3$, and q is the cardinality of $\mathbf{F}_p[X]/(T)$. The parameter s has the same meaning as in Fp_ellcard_SEA. If the seadata package is installed, the function will be faster.

GEN FpXQ_ellgroup(GEN a4, GEN a6, GEN N, GEN T, GEN p, GEN *pm) Return the group structure D of the group $E(\mathbf{F}_p[X]/(T))$, which is assumed to be of order N and set *pm to m.

GEN FpXQ_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, GEN p) Returns generators of the group $E(\mathbf{F}_p[X]/(T))$ with the base change ch (see FpXQE_changepoint), where D and m are as returned by FpXQ_ellgroup.

GEN FpXQ_elldivpol(GEN a4, GEN a6, long n, GEN T, GEN p) returns the n-division polynomial of the elliptic curve E.

GEN Fq_elldivpolmod(GEN a4, GEN a6, long n, GEN h, GEN T, GEN p) returns the n-division polynomial of the elliptic curve E modulo the polynomial h.

void FpXQ_elltwist(GEN a4, GEN a6, GEN T, GEN p, GEN *pA4, GEN *pA6) sets *pA4 and *pA6 to the corresponding parameters for the quadratic twist of E.

14.2.12 FpXQE.

GEN FpXQE_changepoint(GEN P, GEN m, GEN a4, GEN T, GEN p) returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a FpXQV).

GEN FpXQE_changepointinv(GEN P, GEN m, GEN a4, GEN T, GEN p) returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a FpXQV).

GEN FpXQE_add(GEN P, GEN Q, GEN a4, GEN T, GEN p)

GEN FpXQE_sub(GEN P, GEN Q, GEN a4, GEN T, GEN p)

GEN FpXQE_dbl(GEN P, GEN a4, GEN T, GEN p)

GEN FpXQE_neg(GEN P, GEN T, GEN p)

GEN FpXQE_mul(GEN P, GEN n, GEN a4, GEN T, GEN p)

GEN random_FpXQE(GEN a4, GEN a6, GEN T, GEN p)

GEN FpXQE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, GEN p) Let G be a point of order o, return e such that $e.P = G$. If e does not exists, the result is currently undefined.

GEN FpXQE_order(GEN P, GEN o, GEN a4, GEN T, GEN p) returns the order of P in the group $E(\mathbf{F}_p[X]/(T))$, where o is a multiple of the order of P, or its factorization.

GEN FpXQE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, GEN p) returns the Tate pairing of the point of m-torsion P and the point Q.

GEN FpXQE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, GEN p) returns the Weil pairing of the points of m-torsion P and Q.

GEN RgE_to_FpXQE(GEN P, GEN T, GEN p) returns the FpXQE obtained by applying Rg_to_FpXQ coefficientwise.

14.3 Functions related to modular polynomials.

Variants of polmodular, returning the modular polynomial of prime level L for the invariant coded by inv (0: j, 1: Weber-f, see polclass for the full list).

GEN polmodular_ZXX(long L, long inv, long vx, long vy) returns a bivariate polynomial in variables vx and vy.

GEN polmodular_ZM(long L, long inv) returns a matrix of (integral) coefficients.

GEN Fp_polmodular_evalx(long L, long inv, GEN J, GEN p, long v, int derivs) returns the modular polynomial evaluated at J modulo the prime p in the variable v (if derivs is nonzero, returns a vector containing the modular polynomial and its first and second derivatives, all evaluated at J modulo p).

14.3.1 Functions related to modular invariants.

void check_modinv(long inv) report an error if inv is not a valid code for a mdular invariant.

int modinv_good_disc(long inv, long D) test whether the invariant inv is defined for the discriminant D.

int modinv_good_prime(long inv, long D) test whether the invariant inv is defined for the prime p.

long modinv_height_factor(long inv) return the height factor of the modular invariant inv with respect to the j-invariant. This is an integer n such that the j-invariant is asymptotically of the order of the n-th power of the invariant inv.

long modinv_is_Weber(long inv) test whether the invariant inv is a power of Weber f.

long modinv_is_double_eta(long inv) test whether the invariant inv is a double η quotient.

long disc_best_modinv(long D) the integer D being a negative discriminant, return the modular invariant compatible with D with the highest height factor.

GEN Fp_modinv_to_j(GEN x, long inv, GEN p) Let Φ the modular equation between j and the modular invariant inv, return y such that $\Phi(y, x) = 0 \pmod p$.

14.4 Other curves.

The following functions deal with hyperelliptic curves in weighted projective space $\mathbf{P}_{(1,d,1)}$, with coordinates (x, y, z) and a model of the form $y^2 = T(x, z)$, where T is homogeneous of degree $2d$, and squarefree. Thus the curve is nonsingular of genus $d - 1$.

`long hyperell_locally_soluble(GEN T, GEN p)` assumes that $T \in \mathbf{Z}[X]$ is integral. Returns 1 if the curve is locally soluble over $\mathbf{Q}_p$, 0 otherwise.

`long nf_hyperell_locally_soluble(GEN nf, GEN T, GEN pr)` let K be a number field, attached to `nf`, `pr` a *prid* attached to some maximal ideal $\mathfrak{p}$; assumes that $T \in \mathbf{Z}_K[X]$ is integral. Returns 1 if the curve is locally soluble over $K_{\mathfrak{p}}$. The argument `nf` is a true *nf* structure.

Chapter 15: L-functions

15.1 Accessors.

long is_linit(GEN data)

GEN ldata_get_an(GEN ldata)

GEN ldata_get_dual(GEN ldata)

long ldata_isreal(GEN ldata)

GEN ldata_get_gammavec(GEN ldata)

long ldata_get_degree(GEN ldata)

GEN ldata_get_k(GEN ldata)

GEN ldata_get_k1(GEN ldata)

GEN ldata_get_conductor(GEN ldata)

GEN ldata_get_rootno(GEN ldata)

GEN ldata_get_residue(GEN ldata)

long ldata_get_type(GEN ldata)

long linit_get_type(GEN linit)

GEN linit_get_ldata(GEN linit)

GEN linit_get_tech(GEN linit)

GEN lfun_get_domain(GEN tech)

GEN lfun_get_dom(GEN tech)

long lfun_get_bitprec(GEN tech)

GEN lfun_get_factgammavec(GEN tech)

GEN lfun_get_step(GEN tech)

GEN lfun_get_pol(GEN tech)

GEN lfun_get_Residue(GEN tech)

GEN lfun_get_k2(GEN tech)

GEN lfun_get_w2(GEN tech)

GEN lfun_get_expot(GEN tech)

long lfun_get_bitprec(GEN tech)

`GEN lfunprod_get_fact(GEN tech)`

`GEN theta_get_an(GEN tdata)`

`GEN theta_get_K(GEN tdata)`

`GEN theta_get_R(GEN tdata)`

`long theta_get_bitprec(GEN tdata)`

`long theta_get_m(GEN tdata)`

`GEN theta_get_tdom(GEN tdata)`

`GEN theta_get_isqrtN(GEN tdata)`

15.2 Conversions and constructors.

`GEN lfunmisc_to_ldata(GEN obj)` converts `obj` to `Ldata` format. Exception if `obj` cannot be converted.

`GEN lfunmisc_to_ldata_shallow(GEN obj)` as `lfunmisc_to_ldata`, shallow result. Exception if `obj` cannot be converted.

`GEN lfunmisc_to_ldata_shallow_i(GEN obj)` as `lfunmisc_to_ldata_shallow`, returning `NULL` on failure.

`GEN lfunrtopoles(GEN r)`

`int sdomain_isincl(double k, GEN dom, GEN dom0)`

`GEN ldata_vecan(GEN ldata, long N, long prec)` return the vector of coefficients of indices 1 to N to precision `prec`. The output is allowed to be a `t_VECSMALL` when the coefficients are known to be all integral and fit into a `long`; for instance the Dirichlet L function of a real character or the L-function of a rational elliptic curve.

`GEN ldata_newprec(GEN ldata, long prec)` return a shallow copy of `ldata` with fields accurate to precision `prec`.

`long etaquotype(GEN *peta, GEN *pN, GEN *pk, GEN *pCHI, long *pv, long *psd, long *pcusp)` Let `eta` be the integer matrix factorization supposedly attached to an η-quotient $f(z) = \prod_i \eta(n_i z)^{e_i}$. Assuming `*peta` is initially set to `eta`, this function returns 0 if there is a type error or this does not define a function on some $X_0(N)$. Else it returns 1 and sets

- `*peta` to a normalized factorization (as would be returned by `factor`),
- `*pN` to the level N of f,
- `*pk` to the modular weight k of f,
- `*pCHI` to the Nebentypus of f (quadratic character) as an integer,
- `*pv` to the valuation at infinity $v_q(f)$,
- `*psd` to 1 if and only if f is self-dual,
- `*pcusp` to 1 if f is cuspidal, else to 0 if f holomorphic at all cusps, else to -1.

The last three arguments `pCHI`, `pv` and `pcusp` can be set to `NULL`, in which case the relevant information is not computed, which saves time.

15.3 Variants of GP functions.

`GEN lfun(GEN ldata, GEN s, long bitprec)`

`GEN lfuninit(GEN ldata, GEN dom, long der, long bitprec)`

`GEN lfuninit_make(long t, GEN ldata, GEN tech, GEN domain)`

`GEN lfunlambda(GEN ldata, GEN s, long bitprec)`

`GEN lfunquadneg(long D, long k)` for $L(\chi_D, k)$, D fundamental discriminant and $k \geq 0$.

`long lfunthetacost(GEN ldata, GEN tdom, long m, long bitprec)`: `lfunthetacost0` when the first argument is known to be an `Ldata`.

`GEN lfunthetacheckinit(GEN data, GEN tinf, long m, long bitprec)`

`GEN lfunrootno(GEN data, long bitprec)`

`GEN lfunzetakinit(GEN nf, GEN dom, long der, long bitprec)` where `nf` is a true *nf* structure.

`GEN lfunellmfpeters(GEN E, long bitprec)`

`GEN ellanalyticrank(GEN E, long prec)` DEPRECATED.

`GEN ellL1(GEN E, long prec)` DEPRECATED.

15.4 Inverse Mellin transforms of Gamma products.

`GEN gammamellininv(GEN Vga, GEN s, long m, long bitprec)`

`GEN gammamellininvinit(GEN Vga, long m, long bitprec)`

`GEN gammamellininvrt(GEN K, GEN s, long bitprec)` no GC-clean, but suitable for `gerepileupto`.

`int Vgaeasytheta(GEN Vga)` return 1 if the inverse Mellin transform is an exponential and 0 otherwise.

`double dbllemma526(double a, double b, double c, long B)`

`double dblcoro526(double a, double c, long B)`

Chapter 16: Modular symbols

`void checkms(GEN W)` raise an exception if W is not an *ms* structure from `msinit`.

`void checkmspadic(GEN W)` raise an exception if W is not an *mspadic* structure from `mspadicinit`.

`GEN mseval2_ooQ(GEN W, GEN phi, GEN c)` let W be a `msinit` structure for $k = 2$, ϕ be a modular symbol with integral values and c be a rational number. Return the integer $\phi(p)$, where p is the path $\{\infty, c\}$.

`void mspadic_parse_chi(GEN s, GEN *s1, GEN *s2)` see `mspadicL`; let χ be the cyclotomic character from $\mathrm{Gal}\big(\mathbf{Q}_p(mu_{p^\infty})/\mathbf{Q}_p\big)$ to $\mathbf{Z}_p^*$ and τ be the Teichmüller character for $p > 2$ and the character of order 2 on $(\mathbf{Z}/4\mathbf{Z})^*$ if $p = 2$. Let s encode the p-adic characther $\chi^s := \langle\chi\rangle^{s_1}\tau^{s_2}$; set `*s1` and `*s2` to the integers s_1 and s_2.

`GEN mspadic_unit_eigenvalue(GEN ap, long k, GEN p, long n)` let p be a prime not dividing the trace of Frobenius `ap`, return the unit root of $x^2 - ap * x + p^(k-1)$ to p-adic accuracy p^n.

Variants of `mfnumcusps` :

`ulong mfnumcuspsu(ulong n)`

`GEN mfnumcusps_fact(GEN fa)` where `fa` is `factor`(n).

`ulong mfnumcuspsu_fact(GEN fa)` where `fa` is `factoru`(n).

Chapter 17: Modular forms

17.1 Implementation of public data structures.

`void checkMF(GEN mf)` raise an exception if the argument is not a modular form space.

`GEN checkMF_i(GEN mf)` return the underlying modular form space if `mf` is either directly a modular form space from `mfinit` or a symbol from `mfsymbol`. Return `NULL` otherwise.

`int checkmf_i(GEN mf)` return 1 if the argument is a modular form and 0 otherwise.

`int checkfarey_i(GEN F)` return 1 if the argument is a Farey symbol (from `mspolygon` or `msfarey`) and 0 otherwise.

17.1.1 Accessors for modular form spaces.

Shallow functions; assume that their argument is a modular form space is created by `mfinit` and checked using `checkMF`.

`GEN MF_get_gN(GEN mf)` return the level N as a `t_INT`.

`long MF_get_N(GEN mf)` return the level N as a `long`.

`GEN MF_get_gk(GEN mf)` return the level k as a `t_INT`.

`long MF_get_k(GEN mf)` return the level k as a `long`.

`long MF_get_r(GEN mf)` assuming the level is a half-integer, return the integer $r = k - (1/2)$.

`GEN MF_get_CHI(GEN mf)` return the nebentypus χ, which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either ± 1 or `t_POLMOD` in t.

`long MF_get_space(GEN mf)` returns the space type, corresponding to `mfinit`'s `space` flag. The current list is

```
mf_NEW, mf_CUSP, mf_OLD, mf_EISEN, mf_FULL
```

`GEN MF_get_basis(GEN mf)` return the $\mathbf{Q}$-basis of the space, concatenation of `MF_get_E` and `MF_get_S`, in this order; the forms have coefficients in $\mathbf{Q}(\chi)$. Low-level version of `mfbasis`.

`long MF_get_dim(GEN mf)` returns the dimension d of the space. It is the cardinality of `MF_get_basis`.

`GEN MF_get_E(GEN mf)` returns a $\mathbf{Q}$-basis for the subspace spanned by Eisenstein series in the space; the forms have coefficients in $\mathbf{Q}(\chi)$.

`GEN MF_get_S(GEN mf)` returns a $\mathbf{Q}$-basis for the cuspidal subspace in the space; the forms have coefficients in $\mathbf{Q}(\chi)$.

GEN MF_get_fields(GEN mf) returns the vector of polynomials defining each Galois orbit of newforms over $\mathbf{Q}(\chi)$. Uses memoization: a first call splits the space and may be costly; subsequent calls return the cached result.

GEN MF_get_newforms(GEN mf) returns a vector vF containing the coordinates of the eigenforms on MF_get_basis (mftobasis form). Low-level version of mfeigenbasis, whose elements are recovered as mflinear(mf, gel(vF,i)). Uses memoization, sharing the same data as MF_get_fields. Note that it is much more efficient to use mfcoefs(mf,) then multiply by this vector than to compute the coefficients of eigenforms from mfeigenbasis individually.

The following accessors are technical,

GEN MF_get_M(GEN mf) the $(1+m) \times d$ matrix whose j-th column contain the coefficients of the j-th entry in MF_get_basis, m is the optimal "Sturm bound" for the space: the maximum of the $v_\infty(f)$ over nonzero forms. It has entries in $\mathbf{Q}(\chi)$.

GEN MF_get_Mindex(GEN mf) is a t_VECSMALL containing d row indices, the corresponding rows of M form an invertible matrix M_0.

GEN MF_get_Minv(GEN mf) the inverse of M_0 in a form suitable for fast multiplication.

GEN MFcusp_get_vMjd(GEN mf) valid only for a full *cuspidal* space. Then the functions in MF_get_S are of the form $B_d T_j Tr_M^{new}$. This returns the vector of triples (t_VECSMALL) $[M, j, d]$, in the same order.

GEN MFnew_get_vj(GEN mf) valid only for a *new* space. Then the functions in MF_get_S are of the form $T_j Tr_N^{new}$. This returns a t_VECSMALL of the Hecke indices j, in the same order.

17.1.2 Accessors for individual modular forms.

GEN mf_get_gN(GEN F) return the level of F, which may be a multiple of the conductor, as a t_INT

long mf_get_N(GEN F) return the level as a long.

GEN mf_get_gk(GEN F) return the weight of F as a t_INT or a t_FRAC with denominator 2 (half-integral weight).

long mf_get_k(GEN F) return the weight as a long; if the weight is not integral, this raises an exception.

long mf_get_r(GEN F) assuming F is a modular form of half-integral weight $k = (2r+1)/2$, return $r = k - (1/2)$.

GEN mf_get_CHI(GEN F) return the nebentypus, which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either ± 1 or t_POLMOD in t.

GEN mf_get_field(GEN F) return the polynomial (in variable y) defining $\mathbf{Q}(f)$ over $\mathbf{Q}(\chi)$.

GEN mf_get_NK(GEN F) return the tag attached to F: a vector containing gN, gk, CHI, field. Never use its component directly, use individual accessors as above.

long mf_get_type(GEN F) returns a symbolic name for the constructur used to create the form, e.g. t_MF_EISEN for a general Eisenstein series. A form has a recursive structure represented by a tree: its definition may involve other forms, e.g. the tree attached to $T_n f$ contains f as a subtree. Such trees have *leaves*, forms which do not contain a strict subtree, e.g. t_MF_DELTA is a leaf, attached to Ramanujan's Δ.

Here is the current list of types; since the names are liable to change, they are not documented at this point. Use `mfdescribe` to visualize their mathematical structure.

```
/*leaves*/
  t_MF_CONST, t_MF_EISEN, t_MF_Ek, t_MF_DELTA, t_MF_ETAQUO, t_MF_ELL,
  t_MF_DIHEDRAL, t_MF_THETA, t_MF_TRACE, t_MF_NEWTRACE,
/*recursive*/
  t_MF_MUL, t_MF_POW, t_MF_DIV, t_MF_BRACKET, t_MF_LINEAR, t_MF_LINEAR_BHN,
  t_MF_SHIFT, t_MF_DERIV, t_MF_DERIVE2, t_MF_TWIST, t_MF_HECKE,
  t_MF_BD,
```

17.1.3 Nebentypus. The characters stored in modular forms and modular form spaces have a special structure. One can recover the parameters of an ordinary Dirichlet character by `G = gel(CHI,1)` (the underlying `znstar`) and `chi = gel(CHI,2)` (the underlying character in `znconreylog` form).

`long mfcharmodulus(GEN CHI)` the modulus of χ.

`long mfcharorder(GEN CHI)` the order of χ.

`GEN mfcharpol(GEN CHI)` the cyclotomic polynomial Φ_n defining $\mathbf{Q}(\chi)$, always normalized so that n is not 2 mod 4.

17.1.4 Miscellaneous functions.

`long mfnewdim(long N, long k, GEN CHI)` dimension of the new part of the cuspidal space.

`long mfcuspdim(long N, long k, GEN CHI)` dimension of the cuspidal space.

`long mfolddim(long N, long k, GEN CHI)` dimension of the old part of the cuspidal space.

`long mfeisensteindim(long N, long k, GEN CHI)` dimension of the Eisenstein subspace.

`long mffulldim(long N, long k, GEN CHI)` dimension of the full space.

`GEN mfeisensteinspaceinit(GEN NK)`

`GEN mfdiv_val(GEN F, GEN G, long vG)`

`GEN mfembed(GEN E, GEN v)`

`GEN mfmatembed(GEN E, GEN v)`

`GEN mfvecembed(GEN E, GEN v)`

`long mfsturmNgk(long N, GEN k)`

`long mfsturmNk(long N, long k)`

`long mfsturm_mf(GEN mf)`

`long mfiscuspidal(GEN mf, GEN F)`

`GEN mftobasisES(GEN mf, GEN F)`

`GEN mftocol(GEN F, long lim, long d)`

`GEN mfvectomat(GEN vF, long lim, long d)`

Chapter 18:
Plots

A PARI_plot canvas is a record of dimensions, with the following fields:

```
long width;  /* window width */
long height; /* window height */
long hunit;  /* length of horizontal 'ticks' */
long vunit;  /* length of vertical 'ticks' */
long fwidth; /* font width */
long fheight;/* font height */
void (*draw)(PARI_plot *T, GEN w, GEN x, GEN y);
```

The `draw` method performs the actual drawing of a `t_VECSMALL` w (rectwindow indices); x and y are `t_VECSMALL`s of the same length and rectwindow $w[i]$ is drawn with its upper left corner at offset $(x[i], y[i])$. No plot engine is available in `libpari` by default, since thie would introduce a dependency on extra graphical libraries. See the files `src/graph/plot*` for basic implementations of various plot engines: `plotsvg` is particularly simple (`draw` is a 1-liner).

`void pari_set_plot_engine(void (*T)(PARI_plot *))` installs the graphical engine T and initializes the graphical subsystem. No routine in this chapter will work without this initialization.

`void pari_kill_plot_engine(void)` closes the graphical subsystem and frees the ressources it occupies.

18.1 Highlevel functions.

Those functions plot $f(E, x)$ for $x \in [a, b]$, using n regularly spaced points (by default).

`GEN ploth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long prec)` draw physically.

`GEN plotrecth(void *E, GEN(*f)(void*, GEN), long w, GEN a, GEN b, ulong flags, long n, long prec)` draw in rectwindow w.

18.2 Function.

`void plotbox(long ne, GEN gx2, GEN gy2)`

`void plotclip(long rect)`

`void plotcolor(long ne, long color)`

`void plotcopy(long source, long dest, GEN xoff, GEN yoff, long flag)`

`GEN plotcursor(long ne)`

`void plotdraw(GEN list, long flag)`

`GEN plothraw(GEN listx, GEN listy, long flag)`

`GEN plothsizes(long flag)`

`void plotinit(long ne, GEN x, GEN y, long flag)`

`void plotkill(long ne)`

`void plotline(long ne, GEN x2, GEN y2)`

`void plotlines(long ne, GEN listx, GEN listy, long flag)`

`void plotlinetype(long ne, long t)`

`void plotmove(long ne, GEN x, GEN y)`

`void plotpoints(long ne, GEN listx, GEN listy)`

`void plotpointsize(long ne, GEN size)`

`void plotpointtype(long ne, long t)`

`void plotrbox(long ne, GEN x2, GEN y2)`

`GEN plotrecthraw(long ne, GEN data, long flags)`

`void plotrline(long ne, GEN x2, GEN y2)`

`void plotrmove(long ne, GEN x, GEN y)`

`void plotrpoint(long ne, GEN x, GEN y)`

`void plotscale(long ne, GEN x1, GEN x2, GEN y1, GEN y2)`

`void plotstring(long ne, char *x, long dir)`

18.2.1 Obsolete functions. These draw directly to a PostScript file specified by a global variable and should no longer be used. Use `plotexport` and friends instead.

`void psdraw(GEN list, long flag)`

`GEN psplothraw(GEN listx, GEN listy, long flag)`

`GEN psploth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long prec)` draw to a PostScript file.

18.3 Dump rectwindows to a PostScript or SVG file.

w, x, y are three `t_VECSMALL`s indicating the rectwindows to dump, at which offsets. If T is `NULL`, rescale with respect to the installed graphic engine dimensions; else with respect to T.

`char* rect2ps(GEN w, GEN x, GEN y, PARI_plot *T)`

`char* rect2ps_i(GEN w, GEN x, GEN y, PARI_plot *T, int plotps)` if `plotps` is 0, as above; else private version used to implement the `plotps` graphic engine (do not rescale, rotate to portrait orientation).

`char* rect2svg(GEN w, GEN x, GEN y, PARI_plot *T)`

18.4 Technical functions exported for convenience.

`void pari_plot_by_file(const char *env, const char *suf, const char *img)` backend used by the `plotps` and `plotsvg` graphic engines.

`void colorname_to_rgb(const char *s, int *r, int *g, int *b)` convert an `X11` colorname to RGB values.

`void color_to_rgb(GEN c, int *r, int *g, int *b)` convert a pari color (`t_VECSMALL` RGB triple or `t_STR` name) to RGB values.

`void long_to_rgb(long c, int *r, int *g, int *b)` split a standard hexadecimal color value `0xfdf5e6` to its rgb components (`0xfd`, `0xf5`, `0xe6`).

Appendix A:
A Sample program and Makefile

We assume that you have installed the PARI library and include files as explained in Appendix A or in the installation guide. If you chose differently any of the directory names, change them accordingly in the Makefiles.

If the program example that we have given is in the file `extgcd.c`, then a sample Makefile might look as follows. Note that the actual file `examples/Makefile` is more elaborate and you should have a look at it if you intend to use `install()` on custom made functions.

```
CC = cc
INCDIR = /home/kb/PARI/pari/../GP/include
LIBDIR = /home/kb/PARI/pari/../GP/lib
CFLAGS = -O -I$(INCDIR) -L$(LIBDIR)

all:  extgcd

extgcd:  extgcd.c
      $(CC) $(CFLAGS) -o extgcd extgcd.c -lpari -lm
```

We then give the listing of the program `examples/extgcd.c` seen in detail in Section 4.10.

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/

/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
  pari_sp av = avma;
  GEN ux = gen_1, vx = gen_0, a = A, b = B;

  if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
  if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
  if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;

    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
  }
  *U = ux;
  *V = diviiexact( subii(a, mulii(A,ux)), B );
  gerepileall(av, 3, &a, U, V); return a;
}

int
```

```
main()
{
  GEN x, y, d, u, v;
  pari_init(1000000,2);
  printf("x = "); x = gp_read_stream(stdin);
  printf("y = "); y = gp_read_stream(stdin);
  d = extgcd(x, y, &u, &v);
  pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
  pari_close();
  return 0;
}
```

Appendix B: PARI and threads

To use PARI in multi-threaded programs, you must configure it using `Configure --enable-tls`. Your system must implement the `__thread` storage class. As a major side effect, this breaks the `libpari` ABI: the resulting library is not compatible with the old one, and `-tls` is appended to the PARI library `soname`. On the other hand, this library is now thread-safe.

PARI provides some functions to set up PARI subthreads. In our model, each concurrent thread needs its own PARI stack. The following scheme is used:

Child thread:

```
void *child_thread(void *arg)
{
  GEN data = pari_thread_start((struct pari_thread*)arg);
  GEN result = ...; /* Compute result from data */
  pari_thread_close();
  return (void*)result;
}
```

Parent thread:

```
pthread_t th;
struct pari_thread pth;
GEN data, result;

pari_thread_alloc(&pth, s, data);
pthread_create(&th, NULL, &child_thread, (void*)&pth); /* start child */
... /* do stuff in parent */
pthread_join(th, (void*)&result); /* wait until child terminates */
result = gcopy(result); /* copy result from thread stack to main stack */
pari_thread_free(&pth); /* ... and clean up */
```

`void pari_thread_valloc(struct pari_thread *pth, size_t s, size_t v, GEN arg)` Allocate a PARI stack of size `s` which can grow to at most `v` (as with `parisize` and `parisizemax`) and associate it, together with the argument `arg`, with the PARI thread data `pth`.

`void pari_thread_alloc(struct pari_thread *pth, size_t s, GEN arg)` As above but the stack cannot grow beyond `s`.

`void pari_thread_free(struct pari_thread *pth)` Free the PARI stack attached to the PARI thread data `pth`. This is called after the child thread terminates, i.e. after `pthread_join` in the parent. Any `GEN` objects returned by the child in the thread stack need to be saved before running this command.

`void pari_thread_init(void)` Initialize the thread-local PARI data structures. This function is called by `pari_thread_start`.

`GEN pari_thread_start(struct pari_thread *t)` Initialize the thread-local PARI data structures and set up the thread stack using the PARI thread data `pth`. This function returns the thread argument `arg` that was given to `pari_thread_alloc`.

`void pari_thread_close(void)` Free the thread-local PARI data structures, but keeping the thread stack, so that a `GEN` returned by the thread remains valid.

Under this model, some PARI states are reset in new threads. In particular

- the random number generator is reset to the starting seed;
- the system stack exhaustion checking code, meant to catch infinite recursions, is disabled (use `pari_stackcheck_init()` to reenable it);
- cached real constants (returned by `mppi`, `mpeuler` and `mplog2`) are not shared between threads and will be recomputed as needed;

The following sample program can be compiled using

```
cc thread.c -o thread.o -lpari -lpthread
```

(Add `-I/-L` paths as necessary.)

```
#include <pari/pari.h> /* Include PARI headers */
#include <pthread.h>   /* Include POSIX threads headers */

void *
mydet(void *arg)
{
  GEN F, M;
  /* Set up thread stack and get thread parameter */
  M = pari_thread_start((struct pari_thread*) arg);
  F = QM_det(M);
  /* Free memory used by the thread */
  pari_thread_close();
  return (void*)F;
}

void *
myfactor(void *arg)  /* same principle */
{
  GEN F, N;
  N = pari_thread_start((struct pari_thread*) arg);
  F = factor(N);
  pari_thread_close();
  return (void*)F;
}

int
main(void)
{
  long prec = DEFAULTPREC;
  GEN M1,M2, N1,N2, F1,F2, D1,D2;
  pthread_t th1, th2, th3, th4; /* POSIX-thread variables */
  struct pari_thread pth1, pth2, pth3, pth4; /* pari thread variables */
```

```
  /* Initialise the main PARI stack and global objects (gen_0, etc.) */
  pari_init(32000000,500000);
  /* Compute in the main PARI stack */
  N1 = addis(int2n(256), 1); /* 2^256 + 1 */
  N2 = subis(int2n(193), 1); /* 2^193 - 1 */
  M1 = mathilbert(149);
  M2 = mathilbert(150);
  /* Allocate pari thread structures */
  pari_thread_alloc(&pth1,8000000,N1);
  pari_thread_alloc(&pth2,8000000,N2);
  pari_thread_alloc(&pth3,32000000,M1);
  pari_thread_alloc(&pth4,32000000,M2);
  /* pthread_create() and pthread_join() are standard POSIX-thread
   * functions to start and get the result of threads. */
  pthread_create(&th1,NULL, &myfactor, (void*)&pth1);
  pthread_create(&th2,NULL, &myfactor, (void*)&pth2);
  pthread_create(&th3,NULL, &mydet,    (void*)&pth3);
  pthread_create(&th4,NULL, &mydet,    (void*)&pth4); /* Start 4 threads */
  pthread_join(th1,(void*)&F1);
  pthread_join(th2,(void*)&F2);
  pthread_join(th3,(void*)&D1);
  pthread_join(th4,(void*)&D2); /* Wait for termination, get the results */
  pari_printf("F1=%Ps\nF2=%Ps\nlog(D1)=%Ps\nlog(D2)=%Ps\n",
              F1,F2, glog(D1,prec),glog(D2,prec));
  pari_thread_free(&pth1);
  pari_thread_free(&pth2);
  pari_thread_free(&pth3);
  pari_thread_free(&pth4); /* clean up */
  return 0;
}
```

Index

SomeWord refers to PARI-GP concepts.
`SomeWord` is a PARI-GP keyword.
SomeWord is a generic index entry.

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