

# Modular forms, modular symbols

(PARI-GP version 2.15.5)

## Modular Forms

### Dirichlet characters

Characters are encoded in three different ways:

- a `t_INT`  $D \equiv 0, 1 \pmod 4$ : the quadratic character  $(D/\cdot)$ ;
- a `t_INTMOD`  $\text{Mod}(m, q)$ ,  $m \in (\mathbf{Z}/q)^*$  using a canonical bijection with the dual group (the Conrey character  $\chi_q(m, \cdot)$ );
- a pair  $[G, \text{chi}]$ , where  $G = \text{znstar}(q, 1)$  encodes  $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \leq k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$  and the vector  $\text{chi} = [c_1, \dots, c_k]$  encodes the character such that  $\chi(g_j) = e(c_j/d_j)$ .

```

initialize G = (Z/qZ)*          G = znstar(q, 1)
convert datum D to [G, chi]    znchar(D)
Galois orbits of Dirichlet characters  chargalois(G)

```

### Spaces of modular forms

Arguments of the form  $[N, k, \chi]$  give the level weight and nebentypus  $\chi$ ;  $\chi$  can be omitted:  $[N, k]$  means trivial  $\chi$ .

```

initialize S_k^new(Gamma_0(N), chi)  mfininit([N, k, chi], 0)
initialize S_k(Gamma_0(N), chi)      mfininit([N, k, chi], 1)
initialize S_k^old(Gamma_0(N), chi)  mfininit([N, k, chi], 2)
initialize E_k(Gamma_0(N), chi)      mfininit([N, k, chi], 3)
initialize M_k(Gamma_0(N), chi)      mfininit([N, k, chi])
find eigenforms                      mfsplit(M)
statistics on self-growing caches     getcache()

```

We let  $M = \text{mfininit}(\dots)$  denote a modular space.

```

describe the space M                mfdescribe(M)
recover (N, k, chi)                 mfparams(M)
... the space identifier (0 to 4)    mfspace(M)
... the dimension of M over C       mfdim(M)
... a C-basis (f_i) of M            mfbasis(M)
... a basis (F_j) of eigenforms     mfeigenbasis(M)
... polynomials defining Q(chi)(F_j)/Q(chi)  mffields(M)

```

```

matrix of Hecke operator T_n on (f_i)  mfheckemat(M, n)
eigenvalues of w_Q                     mfatkineigenvalues(M, Q)
basis of period polynomials for weight k  mferiodpolbasis(k)
basis of the Kohnen +-space           mfkohnenbasis(M)
... new space and eigenforms          mfkohneneigenbasis(M, b)
isomorphism S_k^+(4N, chi) -> S_{2k-1}(N, chi^2)  mfkohnenbijection(M)

```

Useful data can also be obtained a priori, without computing a complete modular space:

```

dimension of S_k^new(Gamma_0(N), chi)  mfdim([N, k, chi])
dimension of S_k(Gamma_0(N), chi)      mfdim([N, k, chi], 1)
dimension of S_k^old(Gamma_0(N), chi)  mfdim([N, k, chi], 2)
dimension of M_k(Gamma_0(N), chi)      mfdim([N, k, chi], 3)
dimension of E_k(Gamma_0(N), chi)      mfdim([N, k, chi], 4)
Sturm's bound for M_k(Gamma_0(N), chi) mfsturm(N, k)

```

### $\Gamma_0(N)$ cosets

```

list of right Gamma_0(N) cosets        mfcosets(N)
identify coset a matrix belongs to    mftocoset

```

### Cusps

```

a cusp is given by a rational number or oo.
lists of cusps of Gamma_0(N)          mfcusps(N)
number of cusps of Gamma_0(N)         mfnnumcusps(N)
width of cusp c of Gamma_0(N)         mfcuspwidth(N, c)
is cusp c regular for M_k(Gamma_0(N), chi)?  mfcuspisregular([N, k, chi], c)

```

## Create an individual modular form

Besides `mfbasis` and `mfeigenbasis`, an individual modular form can be identified by a few coefficients.

modular form from coefficients `mfmbasis(mf, vec)`

There are also many predefined ones:

```

Eisenstein series E_k on Sl_2(Z)      mfeEk(k)
Eisenstein-Hurwitz series on Gamma_0(4)  mfeEH(k)
unary theta function (for character psi)  mfTheta({psi})
Ramanujan's Delta                     mfDelta()
E_k(x)                                 mfeisenstein(k, chi)
E_k(x_1, x_2)                         mfeisenstein(k, chi_1, chi_2)
eta quotient \prod_i eta(a_{i,1} \cdot z)^{a_{i,2}}  mffrometaquo(a)
newform attached to ell. curve E/Q     mffromell(E)
identify an L-function as an eigenform  mffromlfun(L)
theta function attached to Q > 0      mffromqt(Q)
trace form in S_k^new(Gamma_0(N), chi)  mftraceform([N, k, chi])
trace form in S_k(Gamma_0(N), chi)     mfttraceform([N, k, chi], 1)

```

### Operations on modular forms

In this section,  $f, g$  and the  $F[i]$  are modular forms

```

f x g                                  mfmul(f, g)
f/g                                    mfdiv(f, g)
f^n                                    mfpow(f, n)
f(q)/q^v                               mfmshift(f, v)
\sum_{i \leq k} lambda_i F[i], L = [lambda_1, ..., lambda_k]  mflinear(F, L)
f = g?                                 mfishequal(f, g)
expanding operator B_d(f)              mfbfd(f, d)
Hecke operator T_n f                   mfhecke(mf, f, n)
initialize Atkin-Lehner operator w_Q   mfatkininit(mf, Q)
... apply w_Q to f                     mfatkin(w_Q, f)
twist by the quadratic char (D/\cdot)  mftwist(f, D)
derivative wrt. q \cdot d/dq           mfdderiv(f)
see f over an absolute field           mfretoabs(f)
Serre derivative (q \cdot d/dq - k/12) E_2  mfdderivE2(f)
Rankin-Cohen bracket [f, g]_n         mfbracket(f, g, n)
Shimura lift of f for discriminant D   mfmshimura(mf, f, D)

```

### Properties of modular forms

In this section,  $f = \sum_n f_n q^n$  is a modular form in some space  $M$  with parameters  $N, k, \chi$ .

```

describe the form f                    mfdescribe(f)
(N, k, chi) for form f                 mfparams(f)
the space identifier (0 to 4) for f     mfspace(mf, f)
[f_0, ..., f_n]                       mfcocoeffs(f, n)
f_n                                     mfcocoeff(f, n)
is f a CM form?                        mfishCM(f)
is f an eta quotient?                 mfishetaquo(f)
Galois rep. attached to all (1, chi) eigenforms  mfgaloistype(M)
... single eigenform                   mfgaloistype(M, F)
... as a polynomial fixed by Ker rho_F  mfgaloisprojrep(M, F)
decompose f on mfbasis(M)              mftobasis(M, f)
smallest level on which f is defined   mfdconductor(M, f)
decompose f on \oplus S_k^new(Gamma_0(d)), d | N  mftonew(M, f)
valuation of f at cusp c               mfcuspval(M, f, c)
expansion at infinity of f |_k gamma   mflashexpansion(M, f, gamma, n)
n-Taylor expansion of f at i           mftaylor(f, n)
all rational eigenforms matching criteria  mfeigensearch
... forms matching criteria            mfsearch

```

## Forms embedded into C

Given a modular form  $f$  in  $M_k(\Gamma_0(N), \chi)$  its field of definition  $Q(f)$  has  $n = [Q(f) : Q(\chi)]$  embeddings into the complex numbers. If  $n = 1$ , the following functions return a single answer, attached to the canonical embedding of  $f$  in  $\mathbb{C}[[q]]$ ; else a vector of  $n$  results, corresponding to the  $n$  conjugates of  $f$ .

```

complex embeddings of Q(f)             mfembed(f)
... embed coeffs of f                 mfembed(f, v)
evaluate f at tau in H                 mfeval(f, tau)
L-function attached to f               lfunmf(mf, f)
... eigenforms of new space M         lfunmf(M)

```

### Periods and symbols

The functions in this section depend on  $[Q(f) : Q(\chi)]$  as above.

```

initialize symbol fs attached to f      mfsymbol(M, f)
evaluate symbol fs on path p           mfsymboleval(fs, p)
Pettersson product of f and g         mfpetersson(fs, gs)
period polynomial of form f            mfperiodpol(M, f)
period polynomials for eigensymbol FS  mfmanin(FS)

```

## Modular Symbols

Let  $G = \Gamma_0(N)$ ,  $V_k = \mathbf{Q}[X, Y]_{k-2}$ ,  $L_k = \mathbf{Z}[X, Y]_{k-2}$  and  $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ . An element of  $\Delta$  is a *path* between cusps of  $X_0(N)$  via the identification  $[b] - [a] \rightarrow$  path from  $a$  to  $b$ , coded by the pair  $[a, b]$  where  $a, b$  are rationals or  $\infty = (1 : 0)$ .

Let  $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$ ; an element of  $\mathbf{M}_k(G)$  is a  $V_k$ -valued *modular symbol*. There is a natural decomposition  $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$  under the action of the  $*$  involution, induced by complex conjugation. The `msinit` function computes either  $\mathbf{M}_k$  ( $\varepsilon = 0$ ) or its  $\pm$ -parts ( $\varepsilon = \pm 1$ ) and fixes a minimal set of  $\mathbf{Z}[G]$ -generators  $(g_i)$  of  $\Delta$ .

```

initialize M = M_k(Gamma_0(N))^epsilon  msinit(N, k, {epsilon = 0})
the level M                             msgetlevel(M)
the weight k                             msgetweight(M)
the sign epsilon                         msgetsign(M)
Farey symbol attached to G              mspolygon(M)
... attached to H < G                   msfarey(F, inH)
H \ G and right G-action                 mscosets(genG, inH)

```

```

Z[G]-generators (g_i) and relations for Delta  mspathgens(M)
decompose p = [a, b] on the (g_i)           mspathlog(M, p)

```

### Create a symbol

```

Eisenstein symbol attached to cusp c       msfromcusp(M, c)
cuspidal symbol attached to E/Q           msfromell(E)
symbol having given Hecke eigenvalues     msfromhecke(M, v, {H})
is s a symbol?                            msissymbol(M, s)

```

### Operations on symbols

```

the list of all s(g_i)                   mseval(M, s)
evaluate symbol s on path p = [a, b]      mseval(M, s, p)
Pettersson product of s and t            mspetersson(M, s, t)

```

### Operators on subspaces

```

An operator is given by a matrix of a fixed Q-basis. H, if given, is a stable Q-subspace of M_k(G): operator is restricted to H.
matrix of Hecke operator T_p or U_p      mshecke(M, p, {H})
matrix of Atkin-Lehner w_Q               msatkinlehner(M, Q{H})
matrix of the * involution               msstar(M, {H})

```

## Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its first component is a matrix with integer coefficients whose columns form a  $\mathbf{Q}$ -basis. If  $H$  is a Hecke-stable subspace of  $M_k(G)^+$  or  $M_k(G)^-$ , it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform  $\sum_n a_n q^n$ .

cuspidal subspace $S_k(G)^\varepsilon$	<code>muscuspidal(M)</code>
Eisenstein subspace $E_k(G)^\varepsilon$	<code>mseisenstein(M)</code>
new part of $S_k(G)^\varepsilon$	<code>msnew(M)</code>
split $H$ into simple subspaces (of $\dim \leq d$ )	<code>mssplit(M, H, {d})</code>
dimension of a subspace	<code>msdim(M)</code>
$(a_1, \dots, a_B)$ for attached newform	<code>msqexpansion(M, H, {B})</code>
$\mathbf{Z}$ -structure from $H^1(G, L_k)$ on subspace $A$	<code>mslattice(M, A)</code>

## Overconvergent symbols and $p$ -adic $L$ functions

Let  $M$  be a full modular symbol space given by `msinit` and  $p$  be a prime. To a classical modular symbol  $\phi$  of level  $N$  ( $v_p(N) \leq 1$ ), which is an eigenvector for  $T_p$  with nonzero eigenvalue  $a_p$ , we can attach a  $p$ -adic  $L$ -function  $L_p$ . The function  $L_p$  is defined on continuous characters of  $\text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ ; in GP we allow characters  $\langle \chi \rangle^{s_1} \tau^{s_2}$ , where  $(s_1, s_2)$  are integers,  $\tau$  is the Teichmüller character and  $\chi$  is the cyclotomic character.

The symbol  $\phi$  can be lifted to an *overconvergent* symbol  $\Phi$ , taking values in spaces of  $p$ -adic distributions (represented in GP by a list of moments modulo  $p^n$ ).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that  $v_p(a_p) = 0$  if *flag* = 0 (fastest), and that  $v_p(a_p) \geq \textit{flag}$  otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions  $mu$  attached to  $\Phi$  allowing to compute  $L_p$  to high accuracy.

initialize $Mp$ to lift symbols	<code>mspadicinit(M, p, n, {flag})</code>
lift symbol $\phi$	<code>mstooms(Mp, phi)</code>
eval overconvergent symbol $\Phi$ on path $p$	<code>msomseval(Mp, Phi, p)</code>
$mu$ for $p$ -adic $L$ -functions	<code>mspadicmoments(Mp, S, {D = 1})</code>
$L_p^{(r)}(\chi^s)$ , $s = [s_1, s_2]$	<code>mspadicL(mu, {s = 0}, {r = 0})</code>
$\hat{L}_p(\tau^i)(x)$	<code>mspadicseries(mu, {i = 0})</code>

Based on an earlier version by Joseph H. Silverman

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