L-functions

(PARI-GP version 2.16.1)

Characters

A character on the abelian group $G = \sum_{j\le k}(\mathbf{Z}/d_j\mathbf{Z})\cdot g_j$, e.g. from `znstar(q,1)` $\leftrightarrow (\mathbf{Z}/q\mathbf{Z})^*$ or `bnrinit` $\leftrightarrow \mathrm{Cl}_{\mathfrak{f}}(K)$, is coded by $\chi = [c_1,\dots,c_k]$ such that $\chi(g_j) = e(c_j/d_j)$. Our L-functions consider the attached *primitive* character.

Dirichlet characters $\chi_q(m,\cdot)$ in Conrey labelling system are alternatively concisely coded by `Mod(m,q)`. Finally, a quadratic character $(D/\cdot)$ can also be coded by the integer D.

L-function Constructors

An `Ldata` is a GP structure describing the functional equation for $L(s) = \sum_{n\ge1} a_n n^{-s}$.

- Dirichlet coefficients given by closure $a : N \mapsto [a_1,\dots,a_N]$.
- Dirichlet coefficients $a^*(n)$ for dual L-function L^*.
- Euler factor $A = [a_1,\dots,a_d]$ for $\gamma_A(s) = \prod_i \Gamma_{\mathbf{R}}(s+a_i)$,
- classical weight k (values at s and $k-s$ are related),
- conductor N, $\Lambda(s) = N^{s/2}\gamma_A(s)$,
- root number ε; $\Lambda(a, k-s) = \varepsilon\Lambda(a^*, s)$.
- polar part: list of $[\beta, P_\beta(x)]$.

An `Linit` is a GP structure containing an `Ldata` L and an evaluation *domain* fixing a maximal order of derivation m and bit accuracy (`realbitprecision`), together with complex ranges

- for L-function: $R = [c, w, h]$ (coding $|\Re z - c| \le w$, $|\Im z| \le h$); or $R = [w, h]$ (for $c = k/2$); or $R = [h]$ (for $c = k/2$, $w = 0$).
- for θ-function: $T = [\rho, \alpha]$ (for $|t| \ge \rho$, $|\arg t| \le \alpha$); or $T = \rho$ (for $\alpha = 0$).

Ldata constructors

Riemann ζ	`lfuncreate`(1)
Dirichlet for quadratic char. $(D/\cdot)$	`lfuncreate`(D)
Dirichlet series $L(\chi_q(m,\cdot), s)$	`lfuncreate(Mod(m,q))`
Dedekind ζ_K, $K = \mathbf{Q}[x]/(T)$	`lfuncreate`(bnf), `lfuncreate`(T)
Hecke for χ mod $\mathfrak{f}$	`lfuncreate`$([bnr, \chi])$
Artin L-function	`lfunartin`(nf, gal, M, n)
Lattice Θ-function	`lfunqf`(Q)
From eigenform F	`lfunmf`(F)
Quotients of Dedekind η: $\prod_i \eta(m_{i,1}\cdot\tau)^{m_{i,2}}$	`lfunetaquo`(M)
$L(E,s)$, E elliptic curve	`E = ellinit(...)`
$L(Sym^m E, s)$, E elliptic curve	`lfunsympow(E, m)`
Genus 2 curve, $y^2 + xQ = P$	`lfungenus2`$([P,Q])$
Hypergeometric motive $H(a,b;t)$	`lfunhgm(hgminit(a,b), t)`
dual L function $\hat{L}$	`lfundual`(L)
$L_1 \cdot L_2$	`lfunmul`(L_1, L_2)
L_1 / L_2	`lfundiv`(L_1, L_2)
$L(s-d)$	`lfunshift`(L, d)
$L(s)\cdot L(s-d)$	`lfunshift`$(L, d, 1)$
twist by Dirichlet character	`lfuntwist`(L, χ)
low-level constructor	`lfuncreate`$([a, a^*, A, k, N, eps, poles])$
check functional equation (at t)	`lfuncheckfeq`$(L, \{t\})$
parameters $[N, k, A]$	`lfunparams`(L)

Linit constructors

initialize for L	`lfuninit`$(L, R, \{m=0\})$
initialize for θ	`lfunthetainit`$(L, \{T=1\}, \{m=0\})$
cost of `lfuninit`	`lfuncost`$(L, R, \{m=0\})$
cost of `lfunthetainit`	`lfunthetacost`$(L, T, \{m=0\})$
Dedekind ζ_L, L abelian over a subfield	`lfunabelianrelinit`

L-functions

L is an `Ldata` or an `Linit` (more efficient for many values).

Evaluate

$L^{(k)}(s)$	`lfun`$(L, s, \{k=0\})$
$\Lambda^{(k)}(s)$	`lfunlambda`$(L, s, \{k=0\})$
$\theta^{(k)}(t)$	`lfuntheta`$(L, t, \{k=0\})$
generalized Hardy Z-function at t	`lfunhardy`(L, t)

Zeros

order of zero at $s = k/2$	`lfunorderzero`$(L, \{m=-1\})$
zeros $s = k/2 + it$, $0 \le t \le T$	`lfunzeros`$(L, T, \{h\})$

Dirichlet series and functional equation

$[a_n : 1 \le n \le N]$	`lfunan`(L, N)
Euler factor at p	`lfuneuler`(L, p)
conductor N of L	`lfunconductor`(L)
root number and residues	`lfunrootres`(L)

G-functions

Attached to inverse Mellin transform for $\gamma_A(s)$, $A = [a_1,\dots,a_d]$.

initialize for G attached to A	`gammamellininvinit`(A)
$G^{(k)}(t)$	`gammamellininv`$(G, t, \{k=0\})$
asymp. expansion of $G^{(k)}(t)$	`gammamellininvasymp`$(A, n, \{k=0\})$

Hypergeometric motives (HGM)

Hypergeometric templates

Below, H denotes an hypergeometric template from `hgminit`.

HGM template from $A = (\alpha_j), B = (\beta_k)$	`hgminit`$(A, \{B\})$
... from cyclotomic parameters D, E	`hgminit`$(D, \{E\})$
... from gamma vector	`hgminit`(G)
α and β parameters for H	`hgmalpha`(H)
cyclotomic parameters (D, E) of H	`hgmcyclo`(H)
... for all H of degree n	`hgmbydegree`(n)
gamma vector for H	`hgmgamma`(H)
twist A and B by $1/2$	`hgmtwist`(H)
is H symmetrical at $t = 1$?	`hgmissymmetrical`(H)
parameters $[d, w, [P,T], M]$ for H	`hgmparams`(H)

L-function

Let L be the L-function attached to the hypergeometric motive (H, t).

coefficient a_n of L	`hgmcoef`(H, t, n)
coefficients $[a_1,\dots,a_n]$ of L	`hgmcoefs`(H, t, n)
Euler factor at p	`hgmeulerfactor`(H, t, p)
... and valuation of local conductor	`hgmeulerfactor`$(H, t, p, \&e)$
return L as an `Ldata`	`lfunhgm`(H, t)

Based on an earlier version by Joseph H. Silverman

October 2023 v2.38. Copyright © 2023 K. Belabas

Permission is granted to make and distribute copies of this card provided the copyright and this permission notice are preserved on all copies.

Send comments and corrections to ⟨Karim.Belabas@math.u-bordeaux.fr⟩