

L-functions

(PARI-GP version 2.18.0)

Characters

A character on the abelian group $G = \sum_{j \leq k} (\mathbf{Z}/d_j \mathbf{Z}) \cdot g_j$, e.g. from `znstar(q,1)` $\leftrightarrow (\mathbf{Z}/q\mathbf{Z})^*$ or `bnrinit` $\leftrightarrow \text{Cl}_f(K)$, is coded by $\chi = [c_1, \dots, c_k]$ such that $\chi(g_j) = e(c_j/d_j)$. Our L -functions consider the attached *primitive* character.

Dirichlet characters $\chi_q(m, \cdot)$ in Conrey labelling system are alternatively concisely coded by `Mod(m,q)`. Finally, a quadratic character (D/\cdot) can also be coded by the integer D .

L-function Constructors

An `Ldata` is a GP structure describing the functional equation for $L(s) = \sum_{n \geq 1} a_n n^{-s}$.

- Dirichlet coefficients given by closure $a : N \mapsto [a_1, \dots, a_N]$.
- Dirichlet coefficients $a^*(n)$ for dual L -function L^* .
- Euler factor $A = [a_1, \dots, a_d]$ for $\gamma_A(s) = \prod_i \Gamma_{\mathbf{R}}(s + a_i)$,
- classical weight k (values at s and $k - s$ are related),
- conductor N , $\Lambda(s) = N^{s/2} \gamma_A(s)$,
- root number ε ; $\Lambda(a, k - s) = \varepsilon \Lambda(a^*, s)$.
- polar part: list of $[\beta, P_\beta(x)]$.

An `Linit` is a GP structure containing an `Ldata` L and an evaluation *domain* fixing a maximal order of derivation m and bit accuracy (`realbitprecision`), together with complex ranges

- for L -function: $R = [c, w, h]$ (coding $|\Re z - c| \leq w$, $|\Im z| \leq h$); or $R = [w, h]$ (for $c = k/2$); or $R = [h]$ (for $c = k/2$, $w = 0$).
- for θ -function: $T = [\rho, \alpha]$ (for $|t| \geq \rho$, $|\arg t| \leq \alpha$); or $T = \rho$ (for $\alpha = 0$).

Ldata constructors

Riemann ζ	<code>lfuncreate(1)</code>
Dirichlet for quadratic char. (D/\cdot)	<code>lfuncreate(D)</code>
Dirichlet series $L(\chi_q(m, \cdot), s)$	<code>lfuncreate(Mod(m,q))</code>
Dedekind ζ_K , $K = \mathbf{Q}[x]/(T)$	<code>lfuncreate(bnf)</code> , <code>lfuncreate(T)</code>
Hecke for $\chi \bmod \mathfrak{f}$	<code>lfuncreate([bnr, \chi])</code>
Artin L -function	<code>lfunartin(nf, gal, M, n)</code>
Lattice Θ -function	<code>lfunqf(Q)</code>
From eigenform F	<code>lfunmf(F)</code>
Quotients of Dedekind $\eta: \prod_i \eta(m_{i,1} \cdot \tau)^{m_{i,2}}$	<code>lfunetaquo(M)</code>
$L(E, s)$, E elliptic curve	<code>E = ellinit(...)</code>
$L(\text{Sym}^m E, s)$, E elliptic curve	<code>lfunsympow(E, m)</code>
Genus 2 curve, $y^2 + xQ = P$	<code>lfungenus2([P, Q])</code>
Hypergeometric motive $H(a, b; t)$	<code>lfunhgm(hgminit(a,b), t)</code>

dual L function \hat{L}	<code>lfundual(L)</code>
$L_1 \cdot L_2$	<code>lfunmul(L1, L2)</code>
L_1/L_2	<code>lfundiv(L1, L2)</code>
$L(s - d)$	<code>lfunshift(L, d)</code>
$L(s) \cdot L(s - d)$	<code>lfunshift(L, d, 1)</code>
twist by Dirichlet character	<code>lfuntwist(L, \chi)</code>

low-level constructor	<code>lfuncreate([a, a*, A, k, N, eps, poles])</code>
check functional equation (at t)	<code>lfuncheckfeq(L, {t})</code>
parameters $[N, k, A]$	<code>lfunparams(L)</code>

Linit constructors

initialize for L	<code>lfuninit(L, R, {m = 0})</code>
initialize for θ	<code>lfunthetainit(L, {T = 1}, {m = 0})</code>
cost of <code>lfuninit</code>	<code>lfuncost(L, R, {m = 0})</code>
cost of <code>lfunthetainit</code>	<code>lfunthetacost(L, T, {m = 0})</code>
Dedekind ζ_L , L abelian over a subfield	<code>lfunabelianreinit</code>

L-functions

L is an `Ldata` or an `Linit` (more efficient for many values).

Evaluate

$L^{(k)}(s)$	<code>lfun(L, s, {k = 0})</code>
$\Lambda^{(k)}(s)$	<code>lfunlambda(L, s, {k = 0})</code>
$\theta^{(k)}(t)$	<code>lfuntheta(L, t, {k = 0})</code>

generalized Hardy Z -function at t

`lfunhardy(L, t)`

Zeros

order of zero at $s = k/2$	<code>lfunorderzero(L, {m = -1})</code>
zeros $s = k/2 + it$, $0 \leq t \leq T$	<code>lfunzeros(L, T, {h})</code>

Dirichlet series and functional equation

$[a_n: 1 \leq n \leq N]$	<code>lfunan(L, N)</code>
Euler factor at p	<code>lfuneuler(L, p)</code>
conductor N of L	<code>lfunconductor(L)</code>
root number and residues	<code>lfunrootres(L)</code>

G-functions

Attached to inverse Mellin transform for $\gamma_A(s)$, $A = [a_1, \dots, a_d]$.
 initialize for G attached to A `gammamellininivit(A)`
 $G^{(k)}(t)$ `gammamellininiv(G, t, {k = 0})`
 asymp. expansion of $G^{(k)}(t)$ `gammamellininvasymp(A, n, {k = 0})`

Hypergeometric motives (HGM)

Hypergeometric templates

Below, H denotes an hypergeometric template from `hgminit`.
 HGM template from $A = (\alpha_j)$, $B = (\beta_k)$ `hgminit(A, {B})`
 ... from cyclotomic parameters D, E `hgminit(D, {E})`
 ... from gamma vector `hgminit(G)`
 α and β parameters for H `hgmalph(H)`
 cyclotomic parameters (D, E) of H `hgmcyclo(H)`
 ... for all H of degree n `hgmbdegree(n)`
 gamma vector for H `hgmgamma(H)`
 twist A and B by $1/2$ `hgmtwist(H)`
 is H symmetrical at $t = 1$? `hgmissymmetrical(H)`
 parameters $[d, w, [P, T], M]$ for H `hgmparams(H)`

L-function

Let L be the L -function attached to the hypergeometric motive (H, t) .
 coefficient a_n of L `hgcoef(H, t, n)`
 coefficients $[a_1, \dots, a_n]$ of L `hgcoef(H, t, n)`
 Euler factor at p `hgmeulerfactor(H, t, p)`
 ... and valuation of local conductor `hgmeulerfactor(H, t, p, &e)`
 return L as an `Ldata` `lfunhgm(H, t)`

Based on an earlier version by Joseph H. Silverman

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